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Heterotic/Type II Triality and Instantons on K_3

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Abstract: A detailed understanding of instanton effects for half-BPS couplings is pursued in theories with 16 supersymmetries. In particular, we investigate the duality between heterotic string on T^4 and type IIA on K_3 at the $T^4/\mathbb{Z}_2$ orbifold point, as well as their higher and lower dimensional versions. We present a remarkably clean quantitative test of the duality at the level of F^4 couplings, by completely matching a purely one-loop heterotic amplitude to a purely tree-level type II result. The triality of $SO(4,4)$ and several other miracles are shown to be crucial for the duality to hold. Exact non-perturbative new results for type I', F on K_3, M on K_3, and IIB on K_3 are found, and the general form of D-instanton contributions in type IIA or B on $T^4/\mathbb{Z}_2$ is obtained. We also analyze the NS5-brane contributions in type II on $K_3 \times T^2$, and predict the value $\mu(N) = \sum_{d|N}(1/d^3)$ for the bulk contribution to the index of the NS5-brane world-volume theory on $K_3 \times T^2$.

Keywords: String Duality, M-theory, Nonperturbative Effects.

Contents

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1. Introduction

1.1 Instanton effects and BPS saturated couplings

Understanding the rules of instanton calculus in string theory has been a challenging goal over the last few years, the achievement of which has become conceivable thanks to the D-brane description of string solitons [1]. Yet, it is still a difficult problem to compute the contribution of D-instantons to a generic amplitude, not to mention that of NS5-brane instantons relevant for four-dimensional physics. The study of exact BPS saturated amplitudes (R^4 couplings being the primary example) in a weak coupling expansion has shed a welcome light on this problem, and indeed has allowed the first experimental determination of the half-BPS D-instanton measure in the type IIB theory [2], before the latter was derived from first principles [3, 4], together with the leading perturbative corrections in the instanton background [5]. U-duality [6] (see [7] for a review) has been a prominent tool in generalizing these results to toroidal compactifications of M-theory [8, 9, 10], and there is by now a fairly complete understanding of half-BPS instanton effects in these theories [11, 10], modulo still mysterious effects superficially of order e^{-1/g_s^2} [9, 10].

The situation in theories with lower supersymmetry is however not so well understood, even in the case of half-BPS amplitudes in theories with 16 supersymmetries.

The canonical examples in that case correspond to R^2 couplings, believed to be one-loop exact on the type II side, and F^4 couplings, believed to be one-loop exact on the heterotic side [12, 13, 14]. These non-renormalization conjectures are supported by anomaly cancellation arguments and decoupling between the gravitational and vector multiplets. As far as R^2 couplings on the heterotic side are concerned, the only half-BPS instanton is the heterotic 5-brane, and the lack of knowledge of its world-volume dynamics has hindered a direct understanding of its non-perturbative effects on the heterotic side [15, 16], even though interesting results have been obtained on the type I side [17]. We will instead focus on the F^4 couplings, for which several results are already available. On the type I side, there is a quite complete treatment of the D-string instanton contributions [14, 18, 19], even though some ill-understood higher genus contact contributions are needed for the duality to hold [14]. The F^4 couplings on the type I' side have also been computed [20, 21, 22], but the detailed instanton measure remains to be understood. They have also been reproduced from the point of view of F-theory compactified on K_3 at particular singular points of the moduli space [23, 24, 25], but due to the fact that the dilaton is fixed at a finite value, these results give little insight into instanton effects. Finally, closely related four-derivative scalar couplings in the context of type II string theory compactified on K_3 have been obtained [26], which are believed to be related by supersymmetry to F^4 couplings. In the latter case, instanton effects from D-branes wrapped on even homology cycles of K_3 have been identified, and shown to reproduce the type IIB D-instanton contributions in the ten-dimensional decompactification limit. The summation measure was recovered from a D-brane matrix model in [2]. NS5-brane instantons were also found but not thoroughly discussed. It is the purpose of this work to extend these partial results, solve several of the issues raised above, and to try and achieve the same level of understanding as in the maximally supersymmetric case.

1.2 Instantons on K_3 at the orbifold point

In general, a detailed perturbative or instanton computation on a curved manifold like K_3 is hampered by our lack of knowledge of the K_3 stringy geometry beyond simple topological invariants. Our main goal is to obtain a working understanding of D-instanton effects in type II theories compactified on K_3 at the $T^4/\mathbb{Z}_2$ orbifold point of K_3, for which the conformal field theory description is completely solvable but still non-trivial. $\mathbb{Z}_{3,4,6}$ orbifold points are technically more involved but expected to yield similar results. Other solvable descriptions include Gepner points, but those do not in general possess a well defined classical geometry limit, and one must resort to boundary CFT techniques in order to understand these stringy geometries [27]. In the simple $T^4/\mathbb{Z}_2$ orbifold case however, the geometric interpretation is clear, and such techniques can be dispensed with. Instantons simply arise from branes wrapped on even cycles of T^4, or collapsed at the 16 orbifold singularities. They first show

up in type IIB compactified on K_3, or in IIA compactified on $K_3 \times S_1$ where the extra S_1 allows the even D-branes to wrap a Euclidean submanifold. Translating the one-loop heterotic result under the duality map, we shall obtain the contributions of these D-instantons to the half-BPS saturated F^4 amplitudes.

Our method will appear to be equally applicable in any space-time dimension. By going to the appropriate dual description, we will obtained a wealth of complimentary information that we regard as equally interesting. In $D = 6$, we will obtain one of the cleanest tests of heterotic-type IIA duality to our knowledge, by recovering the one-loop result from a type IIA tree-level amplitude. This is arguably the first non-trivial quantitative test of heterotic-type II duality, since all other (with the possible exception of [26], which will be recovered in this work) follow from supersymmetry alone. In $D = 7$, we shall obtain the M-theory four-gluon amplitude for $SU(2)$ gauge bosons located at the A_1 singularities of K_3. In $D = 8$, we shall recover the F^4 amplitude for $SO(8)$ gauge bosons located at the orientifold planes of Sen's F-theory model [28], and amend the existing knowledge [23]. In $D = 9$, we will compute the F^4 couplings at the $SO(16) \times SO(16)$ point, and show that the higher genus contact contributions found in [14] do not arise in this case. Finally, in $D = 4$ we will obtain and analyze the contribution of NS5-brane instanton effects, and extract the corresponding instanton measure. We will also find an interesting non-renormalization property beyond one-loop in the background of the NS5-brane.

1.3 A test of heterotic-type IIA duality

For the convenience of the reader, we would like to sketch the salient points of our analysis in the case of heterotic-type IIA duality in six dimensions, which lies at the basis of our argument and is quite representative of our method. We focus on F^4 couplings involving the 20 gauge fields from the vector multiplets, disregarding the graviphotons for now, and more specifically on the (0,16) of them originating from the Cartan torus of the ten-dimensional gauge group. On the heterotic side, $(\mathrm{Tr}F^2)^2$ couplings related by supersymmetry to Chern-Simons couplings appear at tree-level already. We shall disregard them in this work, since they are analogous to the R^2 couplings and have a trivial dependence on the moduli. More interestingly, the only further contributions to four-gauge-boson F^4 couplings occur at one-loop on the heterotic side, and since they barely saturate the fermionic zero-modes, they are given by the standard integral on the fundamental domain $\mathcal{F}$ of the upper-half plane

$$A^{\mathrm{Het}}_{F^4} = l_{\mathrm{H}}^2 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\bar{Q}^4 \cdot Z_{4,20}(g/l_{\mathrm{H}}^2, b, y)}{\bar{\eta}^{24}} \ . \tag{1.1}$$

Here l_{H} is the heterotic string length, and is reinstated on dimensional grounds. $Z_{4,20}$ denotes the partition function of the heterotic even self-dual lattice of signature (4,20), parameterized by the metric g and Kalb-Ramond field b on the torus T^4

and the Wilson lines y of the 16 $U(1)$ gauge fields in ten dimensions along the 4 circles of the torus T^4. $\bar{Q}^4$ denotes an operator inserting four powers of right-moving momenta in the lattice partition function, depending on the 4 gauge fields considered, and $1/\eta^{24} = 1/q + 24 + \dots$ is the contribution of the 24 right-moving oscillators that generate the Hagedorn density of half-BPS states in the perturbative spectrum of the heterotic string.

Under duality with the type IIA theory compactified on K_3, the six-dimensional string coupling g_6 gets inverted, while the string length is rescaled as $l_s \to g_6 l_s$[2]. Taking into account the particular normalization of the type II Ramond fields, it is easy to see that (1.1) translates into a tree-level type IIA result. On the other hand, it is still given by a modular integral on the fundamental domain of the upper-half plane, which is usually characteristic of one-loop amplitudes. The resolution of this paradox is that on the type IIA side, the gauge fields dual to the $(0,16)$ heterotic ones originate from the twisted sectors of the orbifold: the correlator of four $\mathbb{Z}_2$ twist fields on the sphere can be re-expressed as the correlator of single-valued fields on the double cover of the sphere, which is a torus [29, 30]; its modulus depends on the relative position of the four vertices, and hence should be integrated over. A careful computation yields the tree-level type IIA result

$$A^{\rm IIA}_{F^4} = \frac{1}{g_{\rm II}^2} g_{\rm II}^4 \frac{l_{\rm II}^6}{V_{K_3}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4}(G/l_{\rm II}^2, B) \ , \tag{1.2}$$

where the factors of $g_{\rm II}$ correspond to the tree-level weight and the normalization of the Ramond fields respectively. Here we have focused for simplicity on a particular choice of $(0,16)$ fields: in general, (1.2) involves a shifted lattice sum integrated on a six-fold cover $\mathcal{F}_2$ of the fundamental domain $\mathcal{F}$.

Still this result is not quite of the same form as (1.1). For one thing, the type IIA result, being a half-BPS saturated coupling, does not involve any oscillators, in contrast to the heterotic side. For another, the $[SO(4)\times SO(4)]\backslash SO(4,4,\mathbb{R})$ moduli $G/l_{\rm II}^2, B$ are not the same as the heterotic $g/l_{\rm H}^2, B$. In order to reconcile the two, we need to take several steps:

(i) Moduli identification: The relation between the heterotic and type II moduli can be obtained by studying the BPS spectrum. On the heterotic side, the BPS states are Kaluza-Klein and winding states transforming as a *vector* of $SO(4,4,\mathbb{Z})$, and possibly charged under the 16 $U(1)$ gauge fields. On the type IIA side, a set of BPS states is certainly given by the D0-, D2- and D4-branes wrapped on the even cycles of T^4, which are invariant under the $\mathbb{Z}_2$ involution. These states transform as a *conjugate spinor* of the T-duality group $SO(4,4,\mathbb{Z})$, as D-branes should [7]. We thus find that the heterotic $g/l_{\rm H}^2, b$ and type IIA $G/l_{\rm II}^2, B$ moduli should be related by $SO(4,4)$ *triality* [31], which exchanges the vector and conjugate spinor representations.

[2]In our conventions, we transform the string length but leave the metric invariant. This takes care of the Weyl rescalings needed to go from the various string frames to the Einstein frame.

There are also D2-brane states wrapped on the collapsed spheres at the sixteen orbifold singularities [32], and charged under the corresponding $U(1)$ fields. These are to be identified with the charged BPS states on the heterotic side, and their masses are matched by choosing the Wilson lines as [33]

$$y = \frac{1}{2} \begin{pmatrix} 0101\ 0101\ 0101\ 0101 \\ 0000\ 0000\ 1111\ 1111 \\ 0000\ 1111\ 0000\ 1111 \\ 0011\ 0011\ 0011\ 0011 \end{pmatrix} . \tag{1.3}$$

This can also be derived by realizing that the Wilson lines along the first circle in T^4 map to the B-field fluxes on the collapsed two-spheres, which have been shown to be half a unit in order for the conformal field theory to be non-singular [34]. If we instead put this Wilson line to zero, we recover a gauge symmetry $SO(4)^8 = SU(2)^{16}$, as appropriate for the 16 A_1 singularities of $T^4/\mathbb{Z}_2$. This choice is relevant for M-theory compactified on K_3 at the $\mathbb{Z}_2$ orbifold point. If we further omit the Wilson lines on the 2nd (resp 2nd and 3rd) circles, the gauge symmetry is enlarged to $SO(8)^4$ (resp. $SO(16)^2$), which are relevant for F-theory on K_3 and type I' respectively. These relations explain why our results can easily been applied to these settings as well.

(ii) Hecke identities: At the above choice of Wilson lines, it so happens that the lattice sum simplifies drastically. This phenomenon was noted in a particular example in [23], and we will greatly extend its range of validity. In order to see this, it is useful to reformulate the above choice of Wilson lines on the heterotic T^4 as a $(\mathbb{Z}_2)^4$ freely acting orbifold, so that

$$Z_{4,20} = \frac{1}{2^4} \sum_{h,g} Z_{4,4} \left[{h \atop g}\right] \bar{\Theta}_{16} \left[{h \atop g}\right] \ , \tag{1.4}$$

where g and h run from 0 to 15 and are best seen as four-digit binary numbers; h labels the twisted sector while the summation over g implements the orbifold projection in that sector. The blocks $Z_{4,4} \left[{h \atop g}\right]$ are partition functions of (4,4) lattices with half-integer shifts, and $\bar{\Theta}_{16} \left[{h \atop g}\right]$ are antiholomorphic conformal characters. The operator $\bar{Q}^4$ only acts on the latter. As we shall prove in Appendix A.3, extending techniques first developed in [35], the conformal blocks $\Phi \left[{h \atop g}\right] = Q^4 \Theta_{16} \left[{h \atop g}\right] / \eta^{24}$ occurring in the modular integral can be replaced by two-thirds their image λ under the Hecke operator

$$H_{\Gamma_2^-} . \Phi(\tau) = \frac{1}{2} \left(\Phi\left(-\frac{1}{2\tau}\right) + \Phi\left(\frac{\tau}{2}\right) + \Phi\left(\frac{\tau+1}{2}\right) \right) \tag{1.5}$$

provided this image is a constant real number:

$$H_{\Gamma_2^-} \cdot \left[\frac{Q^4 \Theta_{16} \left[{0 \atop 1}\right]}{\eta^{24}} \right] = \lambda \ . \tag{1.6}$$

We observe that the relation (1.6) holds for all the conformal blocks of interest in this construction. The modular integral thus reduces to

$$A_{F^4}^{\mathrm{Het}} = \frac{2\lambda}{3} l_{\mathrm{H}}^2 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4}(g/l_{\mathrm{H}}^2, b) \tag{1.7}$$

and the Hagedorn density of half-BPS states in (1.4) has thus cancelled. We note that modular integrals such as (1.7) have infrared divergences coming from the vacuum sector in the lattice partition function, and we implicitly subtract the divergent term. This is natural from the point of view of the one-loop heterotic thresholds, and required from the point of view of the tree-level type IIA result since we need to subtract the tree-level exchange of massless modes to get the correction to the two-derivative effective action.

(iii) Triality: the last step needed to identify the type IIA and heterotic result is to understand how triality equates the integrals of the partition function $Z_{4,4}(g/l_{\mathrm{H}}^2, b)$ and $Z_{4,4}(G/l_{\mathrm{II}}^2, B)$ on the fundamental domain of the upper-half plane. It is easy to convince oneself that such an equality cannot hold at the level of integrands, by looking at some decompactification limits. However, it has been shown that such modular integrals could be represented as Eisenstein series for the T-duality group $SO(4,4,\mathbb{Z})$, in the *vector* or (conjugate) *spinor* representations according to one's taste [10]:

$$\pi \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4} = \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{V};s=1} = \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{S};s=1} = \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{C};s=1} \; . \tag{1.8}$$

This implies the invariance of the modular integral of $Z_{4,4}(g/l_s^2, b)$ under triality transformation of the moduli, which completes the argument.

1.4 Outline

The previous discussion was intended as a preview only, and will be made precise and generalized in the rest of the paper. The latter is organized in such a way that the reader may skip the more technical sections without major inconvenience. In Section 2, we give an overview of the various dual descriptions of the heterotic string compactified on a torus $T^d = T^{10-D}$ and derive the precise duality maps involved. Section 3 will be devoted to a derivation of the heterotic F^4 amplitudes and their cousins and their representation in the freely acting orbifold language. In Section 4 we will concentrate on the duality test sketched above, and derive the type IIA tree-level amplitude. Section 5 will be devoted to translating the one-loop heterotic results in $4 \leq D \leq 9$ to their respective dual descriptions, and to interpreting these results as instanton effects. Useful facts involving modular forms and shifted lattice partition functions are gathered in the appendices, together with details on the computation of modular integrals of shifted lattice partition functions.

1.5 Note and Acknowledgements

In the course of this project, we learnt that E. Gava, Narain K. and C. Vafa had tackled this problem independently, and in particular independently noticed that the type II tree-level amplitude was a one-loop result in disguise; we are grateful to Edi Gava for communicating some of their preliminary results. We are also indebted to Stephan Stieberger for his assistance in reconciling our approach with his results with W. Lerche [23]. We also learnt that W. Nahm and K. Wendland independently found the triality of $SO(4,4)$ to be relevant for describing the moduli space of K_3 at the $\mathbb{Z}_2$ orbifold point [36]. We furthermore acknowledge helpful discussions with F. Cachazo, M. Gutperle, W. Lerche and P. Mayr.

2. Moduli identification

In this section, we will discuss how the heterotic string theory can be mapped to its various dual descriptions. We start by briefly recalling some basic results about the heterotic moduli space.

2.1 Toroidal compactifications of the heterotic string

We consider the $E_8 \times E_8$ or $SO(32)$ heterotic string theory compactified on a torus T^d. For $d \leq 5$, the moduli space takes the form

$$\mathbb{R}^+ \times \ [SO(d) \times SO(d+16)] \backslash SO(d,d+16,\mathbb{R})/SO(d,d+16,\mathbb{Z}) \ , \tag{2.1}$$

where the first factor is parameterized by the T-duality invariant dilaton ϕ_{10-d} related to the ten-dimensional heterotic coupling g_{H} by $e^{-2\phi_{10-d}} = V_d/(g_{\mathrm{H}}^2 l_{\mathrm{H}}^d)$, with V_d the volume of the d-torus; the second factor is the standard Narain moduli space, describing the metric g and B-field b of the internal torus, together with the Wilson lines y of the 16 U(1) gauge fields in the Cartan torus of the ten-dimensional gauge group [37]. The right action of the discrete group $SO(d,d+16,\mathbb{Z})$ (by which we mean the automorphism group of the lattice $E_8 \oplus E_8 \oplus H^d$ or $D_{16} \oplus H^d$ depending on the case, where H is the hyperbolic standard lattice) reflects the invariance under T-duality. This moduli space is usually parameterized in the Iwasawa gauge by the $SO(d,d+16,\mathbb{R})$ viel-bein

$$e_{\mathrm{H}} = \begin{pmatrix} v^{-t} & & \\ & 1_{16} & \\ & & v \end{pmatrix} \cdot \begin{pmatrix} 1_d & y & b - yy^t/2 \\ & 1_{16} & -y^t \\ & & 1_d \end{pmatrix} \ , \qquad e_{\mathrm{H}}^t \eta e_{\mathrm{H}} = \eta \ , \qquad \eta = \begin{pmatrix} & & 1_d \\ & 1_{16} & \\ 1_d & & \end{pmatrix} \ , \tag{2.2}$$

where v is the viel-bein of the metric of the internal torus, namely $g = l_{\mathrm{H}}^2 v^t v$. Note in particular that e_{H} depends only on the dimensionless moduli $g/l_{\mathrm{H}}^2, b$ and y. The

right action by the $SO(d,d+k,\mathbb{Z})$ elements

$$\begin{pmatrix} 1_d & y' & -y'y'^t/2 \\ & 1_{16} & -y'^t \\ & & 1_d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1_d & & b' \\ & 1_{16} & \\ & & 1_d \end{pmatrix} \tag{2.3}$$

preserves the Iwasawa gauge and generates the discrete Borel symmetries

$$y \to y + y' \ , \quad b \to b + \frac{1}{2}(y'y^t - yy'^t) \quad \text{or} \quad b \to b + b' \ , \tag{2.4}$$

which should be supplemented by Weyl elements in order to generate the full T-duality group.

In order to determine the mapping of moduli to the dual descriptions, our main strategy will be to compare the BPS mass formula on both sides. For perturbative heterotic BPS states, it is simply given by

$$\mathcal{M}^2 = \frac{1}{l_{\rm H}^2} Q^t (M_{d,d+16} - \eta) Q \ , \tag{2.5}$$

where $Q = (m^i, q^I, n_i)$ is the vector of momenta, charges and windings and $M_{d,d+16} = e_{\rm H}^t e_{\rm H}$ in terms of the viel-bein (2.2). The charges $q^I, I = 1 \dots 16$ take values in the even self dual lattice $E_8 \oplus E_8$ or D_{16}. The degeneracy $d(N)$ of states with $Q^t \eta Q = 2m^i n_i + (q^I)^2 = 2N$ is given by the generating formula

$$\sum d(N) q^N = \frac{1}{\eta^{24}(\tau)} = \frac{1}{q} + 24 + \dots \ , \qquad q = e^{2\pi i \tau} \ . \tag{2.6}$$

This description of the moduli space is quite complete for compactification down to 5 dimensions. For lower dimensional compactification however, the moduli space increases due to the dualization of the NS 2-form into a scalar θ (in four dimensions), or of the 30 $U(1)$ gauge fields into scalars (in three dimensions). As a result, the $\mathbb{R}^+$ factor in (2.1) is enhanced to $U(1)\backslash Sl(2,\mathbb{R})$, parameterized by a complex parameter $S = \theta + i/g_4^2$, acted upon by $Sl(2,\mathbb{Z})$ S-duality transformations [38], whereas in $D=3$ all scalars are unified into a $[SO(8) \times SO(24)]\backslash SO(8,24,\mathbb{R})$ symmetric manifold, acted upon by the U-duality group $SO(8,24,\mathbb{Z})$ [39]. It would be quite interesting to determine $SO(8,24,\mathbb{Z})$ invariant couplings in this case, but we will not attempt to do this here. Instead, we will restrict ourselves to $d \leq 6$, and focus on half-BPS saturated couplings which depend on the heterotic Narain moduli only, and hence receive contributions from one-loop only on the heterotic side.

As motivated in the introduction, we now would like to determine the subspace of the moduli space (2.1) dual to a compactification on a flat space except for possible $\mathbb{Z}_2$ conical singularities. It will turn out that such a description exists only for particular values of the Wilson lines breaking the $SO(32)$ gauge symmetry to a subgroup $SO(2^{5-p})^{p+1}$, $0 \leq p \leq 4$. Choosing p Wilson lines out of the four lines

(1.3) fulfills this condition, and so would of course any permutation of the 16 vertical columns. For $d = 4$, it would seem that any generic value of y breaking the gauge symmetry to $U(1)^{16}$ would do, but this is not correct since y should respect a large discrete group of symmetries that we will discuss in Section 4. It would also seem that this same symmetry breaking pattern (for $p > 0$) could be obtained from $E_8 \times E_8$ heterotic theory: however E_8 cannot be broken to $SO(16)$ by Higgs phenomenon but rather to $SO(14) \times U(1)$, and it is necessary to go to an enhanced symmetry point to recover $SO(16)$ and its subgroups[3]. We therefore restrict ourselves to the more convenient $SO(32)$ heterotic description. In the following we shall also focus on the maximally broken situation $p = min(d, 4)$, since other cases, though interesting, can be obtained by straightforward compactification and have been discussed in [14, 20]. Having fixed the values of y modulo 2, we see that the T-duality group is reduced from $O(d, d+16, \mathbb{Z})$ to $O(d, d, \mathbb{Z})$, or rather to a finite index subgroup of it.

2.2 Type I'

Let us first consider the compactification of the heterotic string on a single circle of radius $R_{\rm H}$ with the Wilson line $y = (0000000011111111)$ breaking the gauge symmetry to $SO(16) \times SO(16)$. This theory admits a dual description as type IIA on the orientifold $S^1/\mathbb{Z}_2$ of a circle of radius $R_{\rm A}$, also known as type I' or IA [40]. The gauge symmetry arises from two groups of eight D8-branes located at each of the fixed points. The mapping between the radii and string length can be most easily obtained by first dualizing the heterotic string to type I on a circle, $(g_s, l_s, R) \to (1/g_s, g_s^{1/2} l_s, R)$, and then T-dualizing to type I'. In this way we get

$$g_{\rm I'} = \frac{l_{\rm H}}{g_{\rm H}^{1/2} R_{\rm H}} \ , \quad l_{\rm I'} = g_{\rm H}^{1/2} l_{\rm H} \ , \quad \frac{R_{\rm I'}}{l_{\rm I'}} = g_{\rm H}^{1/2} \frac{l_{\rm H}}{R_{\rm H}} \ , \tag{2.7}$$

where the quantities on the left-hand side refer to the type I' theory and those on the right-hand side to the heterotic theory. In particular, the heterotic nine-dimensional coupling $g_9 = g_{\rm H}(l_{\rm H}/R_{\rm H})^{1/2}$, parameterizing the $\mathbb{R}^+$ factor in (2.1), becomes $(R_{\rm I'}/l_{\rm I'})^{5/4} g_{\rm I'}^{-3/4}$, so that the factorization of the moduli space does not seem to have a very natural interpretation on the type I' side. Similarly, the mapping of the heterotic Wilson lines is quite involved, and the duality map (2.7) is only correct at the $SO(16) \times SO(16)$ point, to which we shall restrict ourselves. The more general case is discussed in [41], where it is shown that a real version of K_3 underlies the type I' description. The $SO(16) \times SO(16)$ point should then correspond to the $T^4/\mathbb{Z}_2$ orbifold point of K_3.

2.3 F-theory on K_3

We now consider the $SO(32)$ heterotic string compactified on a two-torus of Kähler class $T_{\rm H} = b + iV_{\rm H}$ and complex structure $U_{\rm H}$, at the $SO(8)^4$ point, corresponding

[3]We thank F. Cachazo for explaining this to us.

to a choice of two Wilson lines in (1.3). Following the same reasoning as above, we first dualize to type I on T^2 and then apply a double T-duality on T^2 to go to type IIB on a $T^2/\mathbb{Z}_2$ orientifold, with moduli

$$g_{\rm B} = \frac{l_{\rm H}^2}{V_{\rm H}} \ , \quad l_{\rm B} = g_{\rm H} l_{\rm H}^2 \ , \quad V_{\rm B} = \frac{g_{\rm H}^2 l_{\rm H}^4}{V_{\rm H}} \tag{2.8}$$

and the same complex structure $U_{\rm B} = U_{\rm H}$. This is precisely Sen's construction [28] of F-theory on K_3 [42] at the orbifold point $T^4/\mathbb{Z}_2$, seen as an elliptic fibration over the base $T^2/\mathbb{Z}_2$ with a fiber of complex modulus $U_{\rm F} = a + i/g_{\rm B} = T_{\rm H}$. The real factor in (2.1), corresponding to the heterotic 8D coupling, now parameterizes the size of the base $V_{\rm B}/l_{\rm P}^2$ in 10D Planck units ($l_{\rm P} = g_{\rm B}^{1/4} l_B$), while the $[SO(2)\times SO(18)]\backslash SO(2,18,\mathbb{R})$ moduli parameterize the complex structure of elliptically fibered K_3's. At the $T^4/\mathbb{Z}_2$ orbifold fixed point, an $[SO(2)\times SO(2)]\backslash SO(2,2,\mathbb{R})$ subspace remains available corresponding to the $U_{\rm B}$ and $U_{\rm F}$ moduli, while the remaining 2×16 parameters are fixed at the value of the heterotic Wilson lines. There exists other components in the F-theory moduli space corresponding to a fixed dilaton $U_{\rm F}$ and describing the other $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ orbifold points of K_3 [43, 24], but we shall not describe them here. We simply note that they give rise to exceptional gauge symmetry, and are therefore better accommodated in the heterotic $E_8 \times E_8$ setting; the mapping is then obtained by a further $(T_{\rm H}, U_{\rm H})$ interchange on the heterotic side.

2.4 Type IIA on K_3

The natural next step would be to discuss the dual of heterotic on T^3, namely M-theory on K_3, but we shall find it more convenient to consider heterotic on T^4 and its type IIA dual on K_3 first, before taking the large coupling limit in the next subsection.

We therefore consider the $SO(32)$ heterotic string compactified on a torus T^4 with constant metric g and B-field b, and for now unspecified Wilson lines y. This theory is dual to type IIA compactified on K_3 [44] under the identifications

$$l_{\rm H} = g_{6\rm IIA} l_{\rm II} \ , \quad g_{6\rm IIA} = \frac{1}{g_{6\rm H}} \ , \quad \left(\frac{R_1}{l_{\rm H}}\right)^2 = \frac{V_{K_3}}{l_{\rm II}^4} \ , \tag{2.9}$$

which can be obtained by identifying the IIA NS5-brane on K_3 with the fundamental heterotic string, and the type IIA D0-brane with a heterotic Kaluza-Klein state along the circle of radius R_1 in T^4. This requires breaking the $SO(4,20)$ symmetry to

$SO(3,19)$, and decomposing the viel-bein (2.2) into

$$e_{\rm H} = \begin{pmatrix} \frac{l_{\rm H}}{R_1} & & & & \\ & v_3^{-t} & & & \\ & & 1_{16} & & \\ & & & v_3 & \\ & & & & \frac{R_1}{l_{\rm H}} \end{pmatrix} \cdot \begin{pmatrix} 1 & A & & & \\ & 1_3 & & & \\ & & 1_{16} & & \\ & & & 1_3 & -A^t \\ & & & & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1 & & & b_{13} & 0 \\ & 1_3 & & b_{33} & -b_{13}^t \\ & & 1_{16} & & \\ & & & 1_3 & \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & y_1 & -\frac{y_1 y_3^t}{2} & -\frac{y_1 y_1^t}{2} \\ & 1_3 & y_3 & -\frac{y_3 y_3^t}{2} & -\frac{y_3 y_1^t}{2} \\ & & 1_{16} & -y_3^t & -y_1^t \\ & & & 1_3 & 0 \\ & & & & 1 \end{pmatrix} , \tag{2.10}$$

where v_3, b_{33} are the 3-torus viel-bein and B-field, A and b_{31} are the off-diagonal metric $g^{1i} g_{11}$ and B-field b_{i1} ($i=2,3,4$); y_1 is the Wilson line around the first circle and y_3 are the three Wilson lines around T^3.

On the type IIA side, the $[SO(4)\times SO(20)]\backslash SO(4,20,\mathbb{R})$ moduli space also has a natural decomposition into $\mathbb{R}^+ \times\ [SO(3)\times SO(19)]\backslash SO(3,19,\mathbb{R})$, where the first factor corresponds to the volume of K_3 in type IIA string units, and the second parameterizes the unit volume Einstein metric on K_3 (see [45] for a review). Together with the fluxes of the B-field along the 22 homology 2-cycles, these parameters make up an $SO(4,20)$ matrix

$$e_{\rm IIA} = \begin{pmatrix} \frac{l_{\rm II}^2}{\sqrt{V_{K_3}}} & & \\ & e_{3,19} & \\ & & \frac{\sqrt{V_{K_3}}}{l_{\rm II}^2} \end{pmatrix} \cdot \begin{pmatrix} 1 & B & -\frac{1}{2} B \eta_{3,19} B^t \\ & 1_{22} & -\eta_{3,19} B^t \\ & & 1 \end{pmatrix} , \tag{2.11}$$

where $e_{3,19}$ is the viel-bein parameterizing the Einstein metric of K_3 and $\eta_{3,19}$ denotes the signature $(3,19)$ metric on the space of two-cycles $H_2(K_3)$. This can also be obtained from the BPS mass formula [46]

$$\begin{aligned} \mathcal{M}^2 &= \frac{1}{g_{6\rm IIA}^2 l_{\rm II}^2} q^t (e_{\rm IIA}^t e_{\rm IIA} - \eta) q = \frac{1}{g_{\rm II}^2 l_{\rm II}^2} \left(q_0 - \frac{V_{K_3}}{l_{\rm II}^4} q_4 + B q_2 - \frac{B \eta_{3,19} B^t}{2} q_4 \right)^2 \\ &\quad + \frac{V_{K_3}}{g_{\rm II}^2 l_{\rm II}^6} \left(q_2 - \eta_{3,19} B^t q_4 \right)^t \left(e_{3,19}^t e_{3,19} - \eta_{3,19} \right) \left(q_2 - \eta_{3,19} B^t q_4 \right) , \end{aligned} \tag{2.12}$$

where q_0, q_2 and q_4 denote the D0-,D2- and D4-brane charge. Matching (2.11) with (2.10) gives the last identification in (2.9). The heterotic T-duality group $SO(4,20,\mathbb{Z})$ acting from the right on $e_{\rm IIA}$ is now interpreted as mirror symmetry of the $(4,4)$ K_3 superconformal theory (see [36] for a recent review), while the ADE enhanced symmetries on the heterotic side arise from D2-branes wrapped on vanishing cycles of K_3 on the type II side [47].

We now would like to identify the $T^4/\mathbb{Z}_2$ orbifold point in this moduli space. At that point, we have a very explicit description of the 22 two-cycles in $H_2(K_3)$: 3 self-dual cycles and 3 anti-self-dual cycles come from the 6 two-cycles in $H_2(T^4)$, which are obviously invariant under the $\mathbb{Z}_2$ involution which reverses the sign of the 4 coordinates, while 16 more anti-self-dual ones come from the collapsed two-spheres at any of the 16 singularities. $H_2(T^4,\mathbb{Z})$ has a signature (3,3) even inner product, given by the wedge product of two-forms integrated on $T^4/\mathbb{Z}_2$, and carries a natural metric $M_{3,3}=G\wedge G/V_{K_3}$ orthogonal with respect to the inner product (note that it is independent of the volume of K_3), where G is the metric on T^4. This $SO(3,3)$ matrix is an alternative parameterization of the unit-volume metric of T^4 perhaps less familiar than the standard $G/(\det(G))^{1/4}\in Sl(4)$ representation, and is made possible thanks to the isomorphism $SO(3,3)=Sl(4)$. In order to match with the heterotic side, it is useful to rewrite it in the standard form

$$M_{3,3}=\begin{pmatrix}\gamma^{-1} & \gamma^{-1}\beta\\ \beta^t\gamma^{-1} & \gamma-\beta\gamma^{-1}\beta\end{pmatrix}\ ,\quad \gamma=\frac{1}{G^{11}V_{K_3}}\begin{pmatrix}G_{22} & G_{23} & G_{24}\\ G_{23} & G_{33} & G_{34}\\ G_{24} & G_{34} & G_{44}\end{pmatrix}\ ,\tag{2.13a}$$

$$\beta_{12}=G^{14}/G^{11}\ ,\quad \beta_{23}=G^{12}/G^{11}\ ,\quad \beta_{31}=G^{13}/G^{11}\ ,\tag{2.13b}$$

where the matrix $M_{3,3}$ is written in the basis m^{34},m^{42},m^{23}, m^{12},m^{13},m^{14} of $H_2(T^4)$, and γ (resp. β) are symmetric (resp. antisymmetric) 3×3 matrices. G_{ij} is the metric on T^4, and G^{ij} the inverse metric. Decomposing the 3+16+3 B-fluxes into $B_3, B_{16}, B_{\bar 3}$, we arrive at our final parameterization of the moduli matrix at the $T^4/\mathbb{Z}_2$ orbifold point,

$$e_{\rm IIA}=\begin{pmatrix}\frac{l_{\rm II}^2}{\sqrt{V_{K_3}}} &&&&\\ & u^{-t} &&&\\ && 1_{16} &&\\ &&& u &\\ &&&& \frac{\sqrt{V_{K_3}}}{l_{\rm II}^2}\end{pmatrix}\cdot\begin{pmatrix}1 & B_3 &&&\\ & 1_3 &&&\\ && 1_{16} &&\\ &&& 1_3 & -B_3^t\\ &&&& 1\end{pmatrix}\cdot$$
$$\cdot\begin{pmatrix}1 &&& B_{\bar 3} & 0\\ & 1_3 && \beta & -B_{\bar 3}^t\\ && 1_{16} &&\\ &&& 1_3 &\\ &&&& 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & B_{16} & -\frac{B_{16}\zeta^t}{2} & -\frac{B_{16}B_{16}^t}{2}\\ & 1_3 & \zeta & -\frac{\zeta\zeta^t}{2} & -\frac{\zeta B_{16}^t}{2}\\ && 1_{16} & -\zeta^t & -B_{16}^t\\ &&& 1_3 & 0\\ &&&& 1\end{pmatrix}\ ,\tag{2.14}$$

where u is again a viel-bein for the metric γ, i.e. $u^tu=\gamma$. It is now straightforward to identify the heterotic and type IIA moduli (2.2) and (2.11), and obtain the complete duality map,

$$V_{K_3}=R_1^2\ ,\quad \gamma=g_3\ ,\quad \beta=b_{33}\ ,\quad B_3=A\ ,\tag{2.15a}$$

$$B_{16}=y_1\ ,\quad B_{\bar 3}=b_{13}\ ,\quad \zeta=y_3\tag{2.15b}$$

in respective string units. Forgetting for the moment the Wilson line moduli, what we have obtained here is the triality mapping between the $SO(4,4)$ matrices in the vector representation, as appropriate for the heterotic side whose BPS states transform as a vector of $SO(4,4)$, to the conjugate spinor representation, under which the D-brane BPS states of type IIA on the untwisted cycles of $T^4/\mathbb{Z}_2$ transform. In order to appreciate this, it is useful to consider rectangular tori with vanishing B-field, in which case the mapping reduces to

$$\begin{pmatrix} \ln R_1^{\rm H} \\ \ln R_2^{\rm H} \\ \ln R_3^{\rm H} \\ \ln R_4^{\rm H} \end{pmatrix} = P \cdot \begin{pmatrix} \ln R_1^{\rm IIA} \\ \ln R_2^{\rm IIA} \\ \ln R_3^{\rm IIA} \\ \ln R_4^{\rm IIA} \end{pmatrix} \;, \quad P = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \;, \quad P^2 = 1 \;, \qquad (2.16)$$

where P acts as triality on the Cartan torus of $SO(4,4)$, mapping the conjugate spinor $\boldsymbol{C}$ to the vector representation $\boldsymbol{V}$. We also note that the triality acts on the charges of these representations according to

$$(m_1; m_2, m_3, m_4; n^2, n^3, n^4; n^1) = (m; m^{34}, m^{42}, m^{23}; m^{12}, m^{13}, m^{14}; m^{1234}) \;. \qquad (2.17)$$

We have however not entirely completed our duty, since we still need to determine the values of the heterotic Wilson lines at the $\mathbb{Z}_2$ orbifold point. For this, we switch off the B-flux B_{16} on the 16 collapsed spheres as well as B_3. Due to the 16 A_1 singularities on the orbifold, the type IIA theory develops an $SU(2)^{16}$ enhanced gauge theory. From (2.15), we see that this amounts to setting the heterotic Wilson line y_1 to zero. The three remaining Wilson lines should therefore be such that they break $SO(32)$ to $SU(2)^{16}$. This is indeed the case for the Wilson lines in (1.3). The choice of the fourth Wilson line may seem arbitrary, but this is not so: on the type IIA side, the orbifold conformal field theory $T^4/\mathbb{Z}_2$ has a discrete symmetry group G generated by the D_4 dihedral discrete symmetry of the four $S^1/\mathbb{Z}_2$ CFT's [48], to be discussed further in Section 4.2. As will become clear in Section 3.2.3, (1.3) is the only choice consistent with this symmetry. We therefore deduce the corresponding values of the blow-up parameters ζ and B-flux B_{16} from (2.15).

Let us briefly discuss the case of the $\mathbb{Z}_3$ orbifold point of K_3. In that case, there are 9 A_2 singularities, so the symmetry group is enhanced to $SU(3)^9$ in the absence of discrete B-flux. This has rank 18, and can therefore only happen at an enhanced symmetry point on the heterotic side. Moreover, $SU(3)^9$ cannot be embedded in $SO(32)$. It can however be embedded in $E_6 \times E_6 \times E_6$ (the exceptional gauge symmetry found in [43] for F-theory on K_3 at the $\mathbb{Z}_3$ orbifold point), which is an enhanced symmetry of the $E_8 \times E_8$ heterotic string. It is thus possible to identify the $T^4/\mathbb{Z}_3$ orbifold point in the K_3 moduli space in an analogous way as we did, but we shall refrain from attempting this here.

2.5 M-theory on K_3

As announced, we now recover the dual description of the heterotic string compactified on T^3 by decompactifying the heterotic circle of radius R_1 in the above description. Since momentum states along this circle are mapped to type IIA D0-branes, this is the limit that takes type IIA on K_3 to M-theory on K_3, with eleven-dimensional Planck length $l_{\rm M}^3 = g_{\rm IIA} l_{\rm II}^3$ [47, 49]. We therefore obtain

$$l_{\rm M}^3 = \frac{g_{\rm H}^2 l_{\rm H}^6}{V_3} \ , \qquad V_{K_3} = \frac{g_{\rm H}^4 l_{\rm H}^{10}}{V_3^2} \ , \tag{2.18}$$

so that the fundamental heterotic string is identified with the M5-brane wrapped on K_3. The $\mathbb{R}^+$ factor in (2.1) now parameterizes the volume of K_3 in eleven-dimensional Planck units,

$$e^{4\phi_7/3} = V_{K_3}/l_{\rm M}^4 \ , \tag{2.19}$$

while the $[SO(3)\times SO(19)]\backslash SO(3,19,\mathbb{R})$ moduli still describe the unit-volume Einstein metric of K_3. At the $T^4/\mathbb{Z}_2$ point, the same parameterization as in (2.11) is valid, restricted to the $SO(3,19)$ subspace:

$$e_{\rm M} = \begin{pmatrix} u^{-t} & & \\ & 1_{16} & \\ & & u \end{pmatrix} \cdot \begin{pmatrix} 1_3 & \zeta & \beta - \frac{\zeta\zeta^t}{2} \\ & 1_{16} & -\zeta^t \\ & & 1_3 \end{pmatrix} \ , \tag{2.20}$$

so that the identification with the heterotic parameters is simply

$$\gamma = g/l_{\rm H}^2 \ , \quad \beta = b \ , \quad \zeta = y \ . \tag{2.21}$$

Whereas the mapping (2.15) could be seen as the statement of triality, the identification (2.21) can be seen as the realization of the exceptional isomorphism $SO(3,3) = Sl(4)$. Note that all the B-field parameters have disappeared, in accordance with the fact that M-theory does not possess any 2-form in its spectrum, nor does K_3 have any three-cycle. In particular, this implies that the 16 singularities are no more resolved by the half-unit B-flux, and therefore a $SU(2)^{16}$ symmetry is expected, arising from the M2-branes wrapped on the collapsed spheres. This is the case if one chooses the Wilson lines y as the last three in (1.3).

2.6 Type IIB on K_3

We now turn to the five-dimensional compactification of the heterotic string on $T^5 = T^4 \times S^1$, with the four Wilson lines (1.3) along T^4, breaking the gauge symmetry to $U(1)^{16}$. This is dual to type IIA on $K_3 \times S^1$ from Section 2.4, but we are interested here in the type IIB description obtained by a further T-duality. Using the standard $R \to l_s^2/R, g \to g l_s/R$ transformation rules, we find

$$g_{\rm 6IIB} = \frac{l_{\rm H}}{R_{\rm H}} \ , \quad l_{\rm II} = l_{\rm H} g_{\rm 6H} \ , \quad R_{\rm B} = \frac{l_{\rm H}^2 g_{\rm 6H}^2}{R_{\rm H}} \ , \tag{2.22}$$

so that in particular the heterotic five-dimensional dilaton is mapped to the size of the type IIB circle in 6D type IIB Planck units,

$$g_{\rm H5} = R_{\rm B}/l_{\rm P} \ , \quad l_{\rm P}^2 = g_{\rm 6IIB} l_{\rm II}^2 \ . \tag{2.23}$$

The $[SO(5) \times SO(21)]\backslash SO(5,21,\mathbb{R})$ moduli, on the other hand, do not involve the circle direction, and actually give the moduli space of the six-dimensional type IIB theory compactified on K_3 only. The full moduli space is obtained from the $[SO(4) \times SO(20)]\backslash SO(4,20,\mathbb{R})$ moduli by adjoining the six-dimensional type IIB coupling, together with the fluxes $\mathcal{B}$ of the Ramond Ramond even forms on the 4+20 even-cycles of K_3:

$$e_{\rm IIB} = \begin{pmatrix} g_{\rm 6IIB} & & \\ & e_{4,20} & \\ & & \frac{1}{g_{\rm 6IIB}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \mathcal{B} & -\frac{1}{2}\mathcal{B}\eta\mathcal{B}^t \\ & 1_{24} & -\eta\mathcal{B}^t \\ & & 1 \end{pmatrix} \ . \tag{2.24}$$

The even forms wrapped on the same 4+20 cycles with two directions less give 4+20 two-form gauge potentials H with self-dual and anti-self-dual field-strength respectively. The right-action by $SO(5,21,\mathbb{Z})$ matrices corresponds to the U-duality symmetry of type IIB on K_3. In Section 5.4, by mapping one-loop F^4 couplings in five-dimensional heterotic string to type IIB on $K_3 \times S_1$ and taking the decompactification limit $R_{\rm B} \to \infty$, we shall be able to derive the exact U-duality invariant $t_{12}H^4$ couplings between 4 self-dual or anti-self-dual two-forms in IIB compactified on K_3, and analyze the resulting instanton contributions.

2.7 Type IIA and IIB on $K_3 \times T^2$

Finally, we want to briefly discuss the duality between the heterotic string on T^6 and type II theories on $K_3 \times T_2$. This duality can be obtained straightforwardly by compactification from the previously discussed ones, and yields the following identifications:

$$S_{\rm H} = T_{\rm IIA} = U_{\rm IIB} \ , \quad T_{\rm H} = S_{\rm IIA} = S_{\rm IIB} \ , \quad U_{\rm H} = U_{\rm IIA} = T_{\rm IIB} \tag{2.25}$$

between the four-dimensional couplings $S = a + i/g_4^2$, Kähler class T and complex structure U of T^2, with the string scales related by $l_{\rm IIA} = l_{\rm IIB} = l_{\rm H}\sqrt{T_{\rm 2H}/S_{\rm 2H}}$. $S_{\rm H}$ and its images parameterize the $U(1)\backslash Sl(2,\mathbb{R})$ part of the moduli space, while $T_{\rm H}$ and $U_{\rm H}$ arise in the decomposition of $SO(6,22)$ into $SO(2,2) \times SO(4,20)$. This will enable us to obtain the NS5-brane instanton contributions to F^4 couplings on the type IIA and B side in Section 5.5, and in particular extract the summation measure in (5.30).

3. Heterotic amplitudes

Having identified the subspace of moduli space dual to $\mathbb{Z}_2$ orbifold in various dimensions, we now would like to compute the one-loop contribution on the heterotic side for half-BPS saturated amplitudes, including the four-derivative couplings

$$t_8 \mathrm{Tr} F^4 \ , \quad t_8 \mathrm{Tr}(F^2)^2 \ , \tag{3.1}$$

where F denotes the field strength of the $d+16$ right-moving gauge bosons or the d left-moving graviphotons, as well as the couplings involving the gravitational sector,

$$t_8 \mathrm{Tr} R^2 \mathrm{Tr} F^2 \ , \quad t_8 (\mathrm{Tr} R^2)^2 \ , \quad t_8 \mathrm{Tr} R^4 \ , \tag{3.2}$$

where t_8 is the familiar eight-index tensor arising in various string amplitudes [50].

Before proceeding with the computation, it is probably worthwhile recalling the arguments supporting the non-renormalization of these couplings beyond one-loop on the heterotic side [12, 13, 14]. First, in ten dimensions these terms are related by supersymmetry to CP-odd couplings such as $B \wedge \mathrm{Tr} F^4$, which should receive no corrections beyond one-loop for anomaly cancellation. A more explicit proof can be given at the level of string amplitudes [12], and goes through in lower dimensions as well [14]. This argument does not apply to the particular combination $t_8 t_8 R^4 = t_8(4\mathrm{Tr} R^4 - (\mathrm{Tr} R^2)^2)$, which forms a superinvariant on its own and could therefore receive higher perturbative corrections. Second, the only heterotic half-BPS instanton is the heterotic 5-brane, which needs a six-cycle to wrap in order to give a finite action instanton effect. For $d < 6$ there can therefore be no non-perturbative contributions beyond the one-loop result. Third, it is consistent with the factorization of the moduli space (2.1) and the T-duality symmetry $O(d, d+16, \mathbb{Z})$ to assume that $t_8 \mathrm{Tr} F^4$ couplings are given at one-loop only and hence independent of the $\mathbb{R}^+$ factor. In $d = 6$ it is plausible that supersymmetry prevents the mixing of the $Sl(2,\mathbb{R})$ dilaton factor with the Narain moduli in F^4 couplings, in the same way as neutral hypermultiplets decouple from vector multiplets in $N = 2$ supergravity, and prevents corrections from NS5-brane instantons [11]. For $d = 7$, U-duality mixes the dilaton with the Narain moduli, so that a similar statement cannot hold. Gauge fields being Poincaré dual to scalars in 3 dimensions, the F^4 couplings translate into four-derivative scalar couplings, and should receive non-perturbative corrections. We will therefore assume that for all $d \leq 6$, the F^4 amplitudes involving four right-moving gauge fields are given at one-loop only on the heterotic side, and disregard a possible tree-level contribution for $t_8 \mathrm{Tr}(F^2)^2$ couplings.

Based on power counting, the F^4 couplings are clearly half-BPS saturated, and the same will appear to be true for their $R^2 F^2$ and R^4 cousins. Indeed, from the point of view of the heterotic world-sheet, space-time supersymmetry arises from the left-moving sector, and gravitons are on the same footing as gauge bosons. This is

not so obvious on the type II side, where part of the gauge bosons arise from the twisted Ramond-Ramond sector while the gravitons come from the untwisted Neveu-Schwarz sector. It has however been argued that R^4 and more generally R^4F^{4g-4} couplings were purely topological for type IIA on K_3, and therefore should be half-BPS saturated as well [52].

For the uncompactified heterotic string, the couplings (3.1),(3.2) have been computed in [53, 54, 55] and shown to involve the zero-modes of the right-moving currents only, reducing to an elliptic genus. It is straightforward to adapt these computations to toroidal compactifications, and in particular to compactifications with discrete Wilson lines as in (1.3). This is what we now discuss, with a particular emphasis on the miraculous simplifications that occur and allow the heterotic-type II duality to hold.

3.1 Orbifold partition function

In order to take advantage of the simple half-integer values of the Wilson lines (1.3), we shall follow [22] and describe the compactification on a torus $T^d(g,b)$ with d Wilson lines from (1.3) as the $(\mathbb{Z}_2)^d$ freely acting orbifold of a torus of double radius by the $\mathbb{Z}_2$ actions which combine a half-period translation on each circle with the corresponding half-integer shift on the lattice. This breaks the $SO(32)$ symmetry to 2^d copies of $SO(2^{5-d})$, as we want. More explicitly, we decompose the partition function of the $(d,d+16)$ lattice as

$$Z_{d,d+16}(g,b,y) = \frac{1}{2^d}\prod_{i=1}^{d}\sum_{h^i,g^i=0}^{1} Z_{d,d}\begin{bmatrix}h^1\dots h^d\\ g^1\dots g^d\end{bmatrix}(g,b)\bar{\Theta}\begin{bmatrix}h^1\dots h^d\\ g^1\dots g^d\end{bmatrix}(0)\ . \tag{3.3}$$

Here, $Z_{d,d}\left[{h \atop g}\right]$ is the T^d lattice partition function, with insertions of $(-)^{m_i g^i}$ and winding shifts $n^i \to n^i + h^i/2$, while $\bar{\Theta}$ is given in terms of the usual θ-functions as

$$\bar{\Theta}\left[{h \atop g}\right](\{v_I\}) = \frac{1}{2}\sum_{a,b=0}^{1}\prod_{\mathrm{d}=0}^{2^d-1}\left(\prod_{I=0}^{2^{4-d}}\theta\begin{bmatrix}a+\mathrm{bin(d)}\cdot h\\ b+\mathrm{bin(d)}\cdot g\end{bmatrix}(v_I^{\mathrm{d}})\right)\ , \tag{3.4}$$

where bin(d) is the d-digit binary representation of d. We have split the 2^5 fermions representing the $SO(32)$ current algebra into 2^d blocks of 2^{5-d} fermions each. The arguments v_I^{d} allow to switch on a gauge background $F_R^I = v_I$ in the I-th direction of the Cartan torus of the d-th copy of the gauge group $SO(2^{5-d})$, and will be useful in deriving the elliptic genus shortly. In particular, for $d=0$ we recover the $SO(32)$ lattice partition function. Setting all the v's to zero, the partition function reads

$$Z_{d,d+16}(g,b,y) = \frac{1}{2^{d+1}}\bar{\theta}_\alpha^{16} Z\left[{0 \atop 0}\right] + \frac{1}{2^d}\left(\bar{\vartheta}_3^8\bar{\vartheta}_4^8 Z\left[{0 \atop \hat{\mathrm{d}}}\right] + \bar{\vartheta}_3^8\bar{\vartheta}_2^8 Z\left[{\hat{\mathrm{d}} \atop 0}\right] + \bar{\vartheta}_2^8\bar{\vartheta}_4^8 Z\left[{\hat{\mathrm{d}} \atop \hat{\mathrm{d}}}\right]\right)\ . \tag{3.5}$$

Here, we have adopted a "modular Einstein convention" whereby $\alpha = 2,3,4$ is summed over all even spin structures and $\mathrm{d} = 0\dots 2^d-1$ is summed over all d-digit

nonnegative numbers, strictly positive if hatted. The three terms in the parenthesis form an orbit of $Sl(2,\mathbb{Z})$, and we will henceforth content ourselves with writing the first term + orb. only. We also drop the d,d subscript on Z when no ambiguity is possible. Note also that thanks to (B.6a), the first unshifted term in (3.5) can be distributed to the three shifted terms if we need to.

3.2 Elliptic genus for higher derivative couplings

We now would like to adapt the computation of [55] to our particular toroidal compactification. Since the amplitude is half-BPS saturated, the left-moving part of the four-gauge-boson (or graviton) amplitude merely provides the kinematic structure, whereas the right-moving currents reduce to their zero-mode part. Focusing on the four-point amplitude for right-moving bosons first, we therefore obtain

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{p_L,p_R\in\Gamma_{d,d+16}} t_8 F_R^I F_R^J F_R^K F_R^L \left(p_R^I p_R^J p_R^K p_R^L - \frac{6}{2\pi\tau_2} p_R^I p_R^J g^{KL} + \frac{3}{4\pi\tau_2^2} g^{IJ} g^{KL} \right) \tau_2^{d/2} \, \frac{q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}}}{\bar{\eta}^{24}} \,, \tag{3.6}$$

where F_R^I stands for the field-strength of any of the $d+16$ right-moving gauge bosons in the Cartan torus of the gauge group. Note in particular that this is expression is both modular invariant and covariant under T-duality. The Dedekind function $1/\bar{\eta}^{24}$ in (3.6) is the contribution of the 24 right-moving oscillators, which generate the tower of perturbative half-BPS states. The integral in (3.6) is actually infrared-divergent and should be regularized. We assume in the following that this is done.

In order to further simplify this expression, we must now distinguish between the $(0,16)$ right-moving bosons coming from the lattice D_{16} and the $(0,d)$ from the torus. In the first case, the insertion of a momentum p_R^I amounts to taking a derivative in (3.4) with respect to the appropriate v_I. The non-holomorphic contributions in (3.6) correct these derivatives $\partial/\partial v$ into modular covariant derivatives $\partial/\partial\hat{v}$. We can therefore omit them and reinstate them at the end by covariance. In the case of the $(0,d)$ gauge bosons, it is more convenient to perform a Poisson resummation on the momenta in the lattice sum: the insertion of p_R^i then amounts to inserting $(m^i - \tau n^i)/\tau_2$, which has modular weight $(0,1)$ as it should. Finally, in the case of gravitons and graviphotons, the analogous statement is that one should allow for a curvature background, thereby inserting a factor $\xi(z) = z\eta^3 e^{-z^2/(8\pi\tau_2)}/\theta_1(z)$ for each pair of space-time coordinates, and take derivatives with respect to z_i for each insertion of a gravi(pho)ton with helicity in the i-th direction. We will quote the results in terms of the integrand Ξ such that the modular integral

$$\Delta_{F^4} = \frac{1}{\mathcal{N}\cdot 2^{d+1}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\bar{\Xi}_{F^4}}{\bar{\eta}^{24}} \tag{3.7}$$

gives the higher-derivative coupling $t_8 \mathrm{Tr} F^4$ in the effective action, where F stands for either a gauge field-strength or the Riemann tensor (seen as an $SO(10-d)$ field-strength), and the trace structure will be made precise. We will denote by F_{d} the field-strength of the right-moving gauge field in the d-th copy of the gauge group, by F_i the right-moving $U(1)$ gauge fields from the torus, and by G_i the left-moving graviphotons. The overall normalization will not be fixed, but we will keep track of the relative normalization of the various couplings, through the combinatorial factor $\mathcal{N}$. We will then make use of the modular identities in Appendix B as well as the theorem proven in Appendix A.4 to simplify these results and bring them in a form appropriate for (i) comparison with the type II tree-level amplitude in Section 4 and (ii) explicit evaluation and simplification for the comparison with the other dual models in Section 5.

3.2.1 $\mathrm{Tr} F_{\mathrm{d}}^4$ and $(\mathrm{Tr} F_{\mathrm{d}}^2)^2$

We start with the four-point amplitude of gauge bosons in a single copy of $SO(2^{5-d})$. For $d < 4$, this is a non-Abelian gauge group, and we must therefore be careful with the identification of the trace structure. Expanding (3.3) to fourth order in the v_I^{d} for fixed d, the $\mathrm{Tr} F_{\mathrm{d}}^4$ combination corresponds to $\sum_I (v_I^{\mathrm{d}})^4$, while $(\mathrm{Tr} F_{\mathrm{d}}^2)^2$ corresponds to $(\sum_I (v_I^{\mathrm{d}})^2)^2$. We thus get

$$\Xi_{\mathrm{Tr} F_{\mathrm{d}_1}^4} = \vartheta_\alpha^{15} \vartheta_\alpha^{''''} Z\left[{0 \atop 0}\right] + \left[\left(\frac{\vartheta_3^{''''}}{\vartheta_3} + \frac{\vartheta_4^{''''}}{\vartheta_4} \right) \vartheta_3^8 \vartheta_4^8 Z\left[{0 \atop \hat{\mathrm{d}}}\right] + \text{ orb.} \right] - 3 \Xi_{(\mathrm{Tr} F_{\mathrm{d}_1}^2)^2} \; , \tag{3.8a}$$

$$\Xi_{(\mathrm{Tr} F_{\mathrm{d}_1}^2)^2} = \vartheta_\alpha^{14} (\vartheta_\alpha^{''})^2 Z\left[{0 \atop 0}\right] + \left[\left(\left(\frac{\vartheta_3^{''}}{\vartheta_3} \right)^2 + \left(\frac{\vartheta_4^{''}}{\vartheta_4} \right)^2 \right) \vartheta_3^8 \vartheta_4^8 Z\left[{0 \atop \hat{\mathrm{d}}}\right] + \text{ orb.} \right] \; , \tag{3.8b}$$

where + orb. denotes the two extra terms obtained from the first by applying S and ST modular transformations. For $d = 4$ and higher, the gauge group is $U(1)$ and there is no distinction between the two structures. Instead, the coupling $(F_{\mathrm{d}_1})^4$ is given by $\Xi_{\mathrm{Tr} F_{\mathrm{d}_1}^4}$ in (3.8a) without the $\Xi_{(\mathrm{Tr} F_{\mathrm{d}_1}^2)^2}$ subtraction. We will discuss in detail the procedure by which we simplify these integrands in the case of the first term in (3.8a), which we define as $\Xi_{(F_{\mathrm{d}_1})^4}$. Other cases can be treated similarly and we will only quote the final result.

Using the summation identity (B.6c), we can write the prefactor of $Z\left[{0 \atop 0}\right]$ as η^{24} while the other $Z\left[{0 \atop 0}\right]$ terms can be combined with the $Z\left[{0 \atop \hat{\mathrm{d}}}\right]$, $Z\left[{\hat{\mathrm{d}} \atop 0}\right]$, and $Z\left[{\hat{\mathrm{d}} \atop \hat{\mathrm{d}}}\right]$ shifted lattice sums to yield projected sums where d runs from 0 to $2^d - 1$:

$$\Xi_{(F_{\mathrm{d}_1})^4} = 96 \eta^{24} Z\left[{0 \atop 0}\right] + \left[\left(\frac{\vartheta_3^{''''}}{\vartheta_3} + \frac{\vartheta_4^{''''}}{\vartheta_4} \right) \vartheta_3^8 \vartheta_4^8 Z\left[{0 \atop \mathrm{d}}\right] + \text{ orb.} \right] \; . \tag{3.9}$$

We can now use the results of Appendix A to compute the integral of (3.9) on the fundamental domain $\mathcal{F}$ of $Sl(2,\mathbb{Z})$. For this, we note that the holomorphic form

η^{24} in the first term cancels against the BPS partition function $1/\eta^{24}$ in (3.7). As far as the terms in brackets is concerned, the integral on $\mathcal{F}$ can be unfolded on the fundamental domain $\mathcal{F}_2^-$ of the Γ_2^- congruence subgroup of $Sl(2,\mathbb{Z})$, a three-fold cover of $\mathcal{F}$, by keeping the first term only. According to the property stated in (A.23), and using (B.7e) we can then replace the modular form $(\vartheta_3^{''''}/\vartheta_3+\vartheta_4^{''''}/\vartheta_4)\vartheta_3^8\vartheta_4^8/\eta^{24}$ by two thirds its value under the Hecke operator (A.18) which turns out to be zero in this case. We finally obtain

$$\Delta_{(F_{\mathrm{d}_1})^4} = \frac{96}{4!\cdot 2^{d+1}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\ , \tag{3.10}$$

which we recall is the complete expression in the Abelian case $d=4$. The fact that all oscillator contributions have cancelled will be crucial for heterotic type II duality to hold, as we will discuss in Section 4.

Moving on to the expression in (3.8b), the manipulations are identical and making use of the summation identity (B.6e) and of the "Hecke" identity (B.7f), yield

$$\Delta_{(\mathrm{Tr}F_{\mathrm{d}_1}^2)^2} = \frac{32}{3\cdot 8\cdot 2^{d+1}}\int_{\mathcal{F}_2^-}\frac{d^2\tau}{\tau_2^2}Z\left[\begin{smallmatrix}0\\\hat{\mathrm{d}}\end{smallmatrix}\right]\ ,\quad (d\leq 3)\ . \tag{3.11}$$

Combining (3.10) and (3.11) together, we thus obtain the full $\mathrm{Tr}F_{\mathrm{d}}^4$ coupling for a non-Abelian gauge group,

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}_1}^4} = \frac{32}{4!\cdot 2^{d+1}}\int_{\mathcal{F}_2^-}\frac{d^2\tau}{\tau_2^2}\left(2Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]-Z\left[\begin{smallmatrix}0\\\mathrm{d}\end{smallmatrix}\right]\right)\ ,\quad (d\leq 3)\ . \tag{3.12}$$

3.2.2 $\mathrm{Tr}F_{\mathrm{d}_1}^2\,\mathrm{Tr}F_{\mathrm{d}_2}^2$

Considering now the coupling between two different gauge groups, we get for $d>1$

$$\begin{aligned}\Xi_{\mathrm{Tr}F_{\mathrm{d}_1}^2\mathrm{Tr}F_{\mathrm{d}_2}^2} =&\vartheta_\alpha^{14}(\vartheta_\alpha^{''})^2 Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\\ &+\left[\left(2\frac{\vartheta_3^{''}}{\vartheta_3}\frac{\vartheta_4^{''}}{\vartheta_4}Z\left[\begin{smallmatrix}00\\\mathrm{d}1\end{smallmatrix}\right]+\left(\left(\frac{\vartheta_3^{''}}{\vartheta_3}\right)^2+\left(\frac{\vartheta_4^{''}}{\vartheta_4}\right)^2\right)Z\left[\begin{smallmatrix}00\\\hat{\mathrm{d}}0\end{smallmatrix}\right]\right)\vartheta_3^8\vartheta_4^8+\text{ orb.}\right]\ .\end{aligned} \tag{3.13}$$

Here, d in the second term runs over the $(d-1)$-digit binary numbers (zero included), whereas $\hat{\mathrm{d}}$ runs over the $(d-1)$-digit binary numbers in the last term (zero excluded). Here we have made a particular choice of gauge fields $F_{\mathrm{d}_1},F_{\mathrm{d}_2}$ corresponding to Wilson lines $y_{\mathrm{d}_1}=\mathrm{bin}(0),y_{\mathrm{d}_2}=\mathrm{bin}(1)$, but the other amplitudes can be obtained by T-duality, and the structure in (3.13) is generic. For $d=1$, the second term does not make sense, and we have instead

$$\Xi_{\mathrm{Tr}F_{\mathrm{d}_1}^2\mathrm{Tr}F_{\mathrm{d}_2}^2} = \vartheta_\alpha^{14}(\vartheta_\alpha^{''})^2 Z_{1,1}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]+\left[2\frac{\vartheta_3^{''}}{\vartheta_3}\frac{\vartheta_4^{''}}{\vartheta_4}Z_{1,1}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\vartheta_3^8\vartheta_4^8+\text{ orb.}\right]\quad (d=1)\ . \tag{3.14}$$

The simplification of expression (3.13) is more involved than the ones of the previous subsection. In this case, we use the summation identity (B.6d) along with (B.4b) to bring the first and last term in the form of the middle one,

$$\Xi_{\mathrm{Tr}F^2_{\mathrm{d}_1}\mathrm{Tr}F^2_{\mathrm{d}_2}} = 16\eta^{24} Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \left[\left(2\frac{\vartheta_3''}{\vartheta_3}\frac{\vartheta_4''}{\vartheta_4} Z\left[\begin{smallmatrix}0\\\mathrm{d}\end{smallmatrix}\right] + \frac{1}{16}\vartheta_2^8 Z\left[\begin{smallmatrix}00\\\hat{\mathrm{d}}0\end{smallmatrix}\right]\right)\vartheta_3^8\vartheta_4^8 + \text{ orb.}\right] . \tag{3.15}$$

Then, using the identity $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$, we see that the first term and the second in the bracket are proportional to η^{24}, while we can use the Hecke identity (B.7g) for the middle term arriving at

$$\Delta_{\mathrm{Tr}F^2_{\mathrm{d}_1}\mathrm{Tr}F^2_{\mathrm{d}_2}} = \frac{16}{3\cdot 4\cdot 2^{d+1}}\int_{\mathcal{F}_2^-}\frac{d^2\tau}{\tau_2^2}\left(Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] - Z\left[\begin{smallmatrix}0\\\mathrm{d}\end{smallmatrix}\right] + 3Z\left[\begin{smallmatrix}00\\\hat{\mathrm{d}}0\end{smallmatrix}\right]\right) \tag{3.16a}$$

$$= \frac{16}{3\cdot 4\cdot 2^{d+1}}\int_{\mathcal{F}_2^-}\frac{d^2\tau}{\tau_2^2}\left(2Z\left[\begin{smallmatrix}0\\\hat{\mathrm{d}}\end{smallmatrix}\right] - 3Z\left[\begin{smallmatrix}00\\\mathrm{d}1\end{smallmatrix}\right]\right) \quad (d>1)\ . \tag{3.16b}$$

Similarly, for the special case $d=1$ in (3.14) we find after analogous steps

$$\Delta_{\mathrm{Tr}F^2_{\mathrm{d}_1}\mathrm{Tr}F^2_{\mathrm{d}_2}} = -\frac{16}{3\cdot 4\cdot 4}\int_{\mathcal{F}_2^-}\frac{d^2\tau}{\tau_2^2} Z_{1,1}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \quad (d=1)\ . \tag{3.17}$$

3.2.3 $F_{\mathrm{d}_1}F_{\mathrm{d}_2}F_{\mathrm{d}_3}F_{\mathrm{d}_4}$

For $d<4$, the gauge lattice partition function (3.4) is even under $v_I^{\mathrm{d}} \to -v_I^{\mathrm{d}}$, so that such a term cannot occur, in agreement with the fact that the generators of $SO(2^{5-d})$ are traceless. For $d=4$ however, it turns out that the coupling between four different $U(1)$ does not vanish, provided the selection rule

$$\mathrm{d}_1+\mathrm{d}_2+\mathrm{d}_3+\mathrm{d}_4 = 0 \mod 16 \tag{3.18}$$

is obeyed, in which case

$$\begin{aligned}\Xi_{F_{\mathrm{d}_1}F_{\mathrm{d}_2}F_{\mathrm{d}_3}F_{\mathrm{d}_4}} =& \eta^{12}(\vartheta_2\vartheta_3\vartheta_4)^4\cdot \\ &\cdot\left(Z\left[\begin{smallmatrix}0100\\1000\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}1000\\0100\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}0100\\1100\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}1000\\1100\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}1100\\0100\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}1100\\1000\end{smallmatrix}\right]\right)\ .\end{aligned} \tag{3.19}$$

Note that the modular orbit now involves six different shifted partition functions. The precise orbit depends on the choice of the four $U(1)$, and we have chosen one example corresponding to $y_{\mathrm{d}_1} = (0000), y_{\mathrm{d}_2} = (0001), y_{\mathrm{d}_3} = (0010), y_{\mathrm{d}_4} = (0011)$. Using the relation $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$, we see that the modular form again cancels against the partition function of the half-BPS states[4], and we are therefore left with

$$\Delta_{F_{\mathrm{d}_1}F_{\mathrm{d}_2}F_{\mathrm{d}_3}F_{\mathrm{d}_4}} = \frac{16}{2^5}\int_{\mathcal{F}_2}\frac{d^2\tau}{\tau_2^2} Z\left[\begin{smallmatrix}0100\\1000\end{smallmatrix}\right] \quad (d=4)\ , \tag{3.20}$$

where the integration is over the six-fold cover $\mathcal{F}_2$ of the fundamental domain $\mathcal{F}$ of $Sl(2,\mathbb{Z})$. As we will see in Section 4.1, the selection rule (3.18) has a direct counterpart in the dual type IIA theory.

[4]The same mechanism was observed in [22].

3.2.4 $\mathrm{Tr}R^4$, $(\mathrm{Tr}R^2)^2$ and $\mathrm{Tr}R^2\mathrm{Tr}F_{\rm d}^2$

We now turn to four-point functions involving gravitons. For a four-graviton amplitude, the elliptic genus [54] is as in the uncompactified heterotic theory, and yields

$$\Xi_{\mathrm{Tr}R^4} = \frac{E_4}{2^7\cdot 3^2\cdot 5}\left[\theta_\alpha^{16} Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + 2\left(\vartheta_3^8\vartheta_4^8 Z\left[\begin{smallmatrix}0\\\hat{\rm d}\end{smallmatrix}\right] + \text{ orb.}\right)\right] \ , \tag{3.21a}$$

$$\Xi_{(\mathrm{Tr}R^2)^2} = \frac{\hat{E}_2^2}{2^9\cdot 3^2}\left[\theta_\alpha^{16} Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + 2\left(\vartheta_3^8\vartheta_4^8 Z\left[\begin{smallmatrix}0\\\hat{\rm d}\end{smallmatrix}\right] + \text{ orb.}\right)\right] \ . \tag{3.21b}$$

For two-graviton two-gauge-boson scattering on the other hand, we need to take two derivatives with respect to $v_{\rm d}$, and we get

$$\Xi_{\mathrm{Tr}R^2\mathrm{Tr}F_{{\rm d}_1}^2} = \frac{\hat{E}_2}{2^3\cdot 3}\left[\vartheta_\alpha^{15}\vartheta_\alpha'' Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \left(\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_4''}{\vartheta_4}\right)\vartheta_3^8\vartheta_4^8 Z\left[\begin{smallmatrix}0\\\hat{\rm d}\end{smallmatrix}\right] + \text{ orb.}\right] \ . \tag{3.22}$$

These amplitudes can all be simplified again by the now familiar method. In particular using (B.6a), (B.7b) we obtain

$$\Delta_{\mathrm{Tr}R^4} = \frac{480}{2^7\cdot 3^2\cdot 5\cdot 2^{d+1}}\int_{\mathcal{F}_2^-}\frac{d^2\tau}{\tau_2^2} Z\left[\begin{smallmatrix}0\\{\rm d}\end{smallmatrix}\right] \ , \tag{3.23a}$$

$$\Delta_{(\mathrm{Tr}R^2)^2} = \frac{96}{2^9\cdot 3^2\cdot 2^{d+1}}\int_{\mathcal{F}_2^-}\frac{d^2\tau}{\tau_2^2} Z\left[\begin{smallmatrix}0\\{\rm d}\end{smallmatrix}\right] \ . \tag{3.23b}$$

It is worthwhile noting that $\Delta_{\mathrm{Tr}R^4} = 4\Delta_{(\mathrm{Tr}R^2)^2}$: this implies that the two terms can be combined into a $t_8t_8R^4 = t_8(4\mathrm{Tr}R^4 - (\mathrm{Tr}R^2)^2)$ coupling, as also arises in type IIA on K_3 [52]. Using (B.6b), (B.7d) in (3.22) we similarly find for the coupling of two gravitons and two right moving $(0,16)$ gauge fields,

$$\Delta_{\mathrm{Tr}R^2\mathrm{Tr}F_{{\rm d}_1}^2} = \frac{16}{2^3\cdot 3\cdot 2^{d+1}}\int_{\mathcal{F}_2^-}\frac{d^2\tau}{\tau_2^2} Z\left[\begin{smallmatrix}0\\{\rm d}\end{smallmatrix}\right] \ . \tag{3.24}$$

3.2.5 $(0,d)$ gauge bosons

As we mentioned above, the insertion of momenta are more easily dealt with in the Lagrangian representation. We thus get

$$\Xi_{F_iF_jF_kF_l} = q_R^i q_R^j q_R^k q_R^l\left[\theta_\alpha^{16} Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + 2\left(\vartheta_3^8\vartheta_4^8 Z\left[\begin{smallmatrix}0\\\hat{\rm d}\end{smallmatrix}\right] + \text{ orb.}\right)\right] \ , \tag{3.25}$$

where q_R^i acts on the torus partition function by inserting a factor of $(m^i + d^i/2 - \tau(n^i + d'^i/2))/\tau_2$ for $Z\left[\begin{smallmatrix}{\rm d}'\\{\rm d}\end{smallmatrix}\right]$. We can also consider the mixed amplitudes of two $(0,d)$ gauge bosons and two gauge bosons or two gravitons respectively, for which

$$\Xi_{(F_iF_j)\mathrm{Tr}R^2} = q_R^i q_R^j\frac{\hat{E}_2}{2^3\cdot 3}\left[\theta_\alpha^{16} Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + 2\left(\vartheta_3^8\vartheta_4^8 Z\left[\begin{smallmatrix}0\\\hat{\rm d}\end{smallmatrix}\right] + \text{ orb.}\right)\right] \ , \tag{3.26a}$$

$$\Xi_{(F_iF_j)\mathrm{Tr}F^2_{\mathrm{d}_1}} = q^i_R q^j_R \left[\vartheta^{15}_\alpha \vartheta''_\alpha Z\left[{}^0_0\right] + \left(\frac{\vartheta''_3}{\vartheta_3} + \frac{\vartheta''_4}{\vartheta_4}\right)\vartheta^8_3\vartheta^8_4 Z\left[{}^0_{\hat{\mathrm{d}}}\right] + \text{ orb.}\right] . \qquad (3.26b)$$

Using the methods described above and (B.6a), (B.7a) or (B.6b), (B.7c), the corresponding couplings are all zero, and we record

$$\Delta_{F_iF_jF_kF_l} = \Delta_{(F_iF_j)\mathrm{Tr}R^2} = \Delta_{(F_iF_j)\mathrm{Tr}F^2_{\mathrm{d}}} = 0 . \qquad (3.27)$$

3.2.6 Graviphotons and summary

We can simply obtain the amplitudes with graviphotons by noting that the graviphoton and graviton vertex operators are similar. We thus obtain four powers of momenta on the holomorphic side, and four on the antiholomorphic side, so that the four graviphoton amplitude starts at eight-derivative order only. In the case of two graviphotons and two right-moving gauge bosons, we obtain two extra powers of momenta from the antiholomorphic side. As a result, the amplitude starts at six-derivative order only. All F^4 effective couplings involving graviphotons thus vanish.

We can summarize the above as follows. At a generic point of the moduli space, and at heterotic one-loop:
(i) The first non-trivial correction to the $(0, d+16)^4$ couplings occurs at the four-derivative level.
(ii) The first non-trivial correction to the $(d, 0)^4$ couplings occurs at the eight-derivative level.
(iii) The first non-trivial correction to the $(d, 0)^2(0, d+16)^2$ couplings occurs at the six-derivative level.

At the $\mathbb{Z}_2$-orbifold point of the six-dimensional heterotic string:
(a) The four-derivative $(0, 16)^4$ couplings have non-vanishing one-loop corrections.
(b) The one-loop four-derivative $(4, 4)^4$ and $(4, 4)^2(0, 16)^2$ couplings are vanishing. The first non-trivial correction for these couplings occurs at the eight-derivative level.

4. Heterotic-type IIA duality in six dimensions

As we already argued in the introduction, the F^4 couplings in the heterotic string on T^4 are given at one-loop only, and translate, through the standard duality map, into a purely tree-level coupling in type IIA on K_3. We can therefore perform a very quantitative test of heterotic-type IIA duality by computing the tree-level F^4 amplitude on the type II side at an orbifold point, where the CFT is exactly soluble. This is the object of the first subsection, the results of which will be summarized and compared to the heterotic side in the second. The reader appalled by the technicalities of Section 4.1 should not feel guilty in proceeding to Section 4.2.

4.1 Type IIA four gauge field amplitude

The 24 gauge fields in type IIA on K_3 originate from the Ramond-Ramond sector. The 4 graviphotons can be understood as the reduction G_i of the 4-form field strength

in $D = 10$ on the 3 self-dual cycles of K_3 together with the ten-dimensional 2-form field-strength G_0, whereas the 20 vector multiplets come from the 4-form in $D = 10$ on the 19 anti-self-dual cycles together with the 6-form field strength K_3 itself: we denote them by F_I and F_0 respectively.

4.1.1 Vertex operators

At the $T^4/\mathbb{Z}_2$ orbifold point, the gauge fields split into untwisted and twisted sectors. The untwisted sector contributes $(4, 4)$ of them, whose vertex operators are simple projections of the ten-dimensional Ramond vertex [56, 57], and can be decomposed into a product of $SO(6)$ and $SO(4)$ spin fields times a ghost part. The $SO(6)$ spin fields $S_\alpha(z), \bar{S}_\alpha(\bar{z})$ have conformal dimension 3/8 and transform as an $SO(6)$ spinor of positive chirality, while S^α and $\bar{S}^\alpha(\bar{z})$ are $SO(6)$ spinors of negative chirality. The $SO(4)$ spin-fields Ψ_a and $\Psi_{\dot{\alpha}}$ involve both chiralities, and are most easily described using the standard dotted notation for $SO(4) \sim SU(2)\times SU(2)$. The ten-dimensional $SO(10)$ spinor decomposes under $SO(4)\times SO(6)$ as $(\mathbf{2}, \mathbf{4}) + (\bar{\mathbf{2}}, \bar{\mathbf{4}})$, while the $SO(10)$ conjugate spinor decomposes as $(\mathbf{2}, \bar{\mathbf{4}}) + (\mathbf{2}, \bar{\mathbf{4}})$. The orbifold projection on T^4 acts on the $SO(4)$ spinors as $(\mathbf{2}, \bar{\mathbf{2}}) \to (\mathbf{2}, -\bar{\mathbf{2}})$. Hence, the vertex operators for untwisted gauge fields read

$$V^m_{-1/2} = e^{-\phi/2} e^{-\tilde{\phi}/2}\ X^m\ S_\alpha \tilde{S}^\beta\ \Sigma^{\mu\nu}{}_\beta^\alpha\ \zeta_{\mu\nu}\ e^{ikX}\ , \tag{4.1a}$$

$$\bar{V}^m_{-1/2} = e^{-\phi/2} e^{-\tilde{\phi}/2}\ \bar{X}^m\ S^\alpha \tilde{S}_\beta\ \Sigma^{\mu\nu}{}_\alpha^\beta\ \zeta_{\mu\nu}\ e^{ikX}\ , \tag{4.1b}$$

where we use the covariant formalism of [56]. $\Sigma^{\mu\nu}_{\alpha\beta}$ are the $SO(6)$ rotation matrices in the spinor representation, and $\zeta_{\mu\nu}$ the polarisation tensors of the field strengths. $e^{-\phi/2}$ is the bosonized superconformal ghost of conformal dimension 3/8. $X^m, \bar{X}^m$ are the fermion combinations

$$X^m = (\Psi^\alpha \epsilon_{\alpha\beta} \tilde{\Psi}^\beta\ , \Psi^\alpha \sigma^{ij}{}_{\alpha\beta} \tilde{\Psi}^\beta)\ , \quad \bar{X}^m = (\Psi^{\dot{\alpha}} \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\Psi}^{\dot{\beta}}\ , \Psi^{\dot{\alpha}} \bar{\sigma}^{ij}_{\dot{\alpha}\dot{\beta}} \tilde{\Psi}^{\dot{\beta}})\ , \tag{4.2}$$

where m runs from 0 to 3 and $\sigma^m = (i1_2, \sigma^i)$. Because of self-duality, only 3 components of σ^{ij} contribute. Together, X^m and $\bar{X}^m$ transform as a conjugate spinor of the T-duality group $SO(4, 4)$. We shall refer to the gauge fields with vertex operators V and $\bar{V}$ as chiral and antichiral respectively. The vertex operators have been displayed in the $(-1/2, -1/2)$ ghost picture, as appropriate for a tree-level four-point amplitude. The 16 remaining gauge bosons come from the 16 twisted sectors, and their vertex operators involve twist fields H^I, $I = 1 \ldots 16$ of conformal dimension 1/4,

$$V^T_{-1/2} = e^{-\phi/2} e^{-\tilde{\phi}/2}\ H^I\ S^\alpha \tilde{S}_\beta\ \Sigma^{\mu\nu}{}_\alpha^\beta\ \zeta_{\mu\nu}\ . \tag{4.3}$$

We have omitted the momentum part, since the Ramond-Ramond vertex operators couple to the world-sheet only through their field-strength, which already provides

the 4 necessary derivatives. We now consider a tree-level amplitude with four vertex operators inserted at $0, x, 1, \infty$ on the complex plane. The correlator factorizes into the ghost part,

$$\langle e^{-\phi/2}(\infty) e^{-\phi/2}(1)\ e^{-\phi/2}(x) e^{-\phi/2}(0)\rangle = [x(1-x)]^{-1/4} \ , \tag{4.4}$$

a 6D spin field part,

$$\langle S_\alpha(\infty) S_\beta(1)\ S_\gamma(x) S_\delta(0)\rangle = [x(1-x)]^{-1/4} \epsilon_{\alpha\beta\gamma\delta} \ , \tag{4.5a}$$

$$\langle S_\alpha(\infty) S^\beta(1)\ S_\gamma(x) S^\delta(0)\rangle = \left[\delta_\alpha^\beta \delta_\gamma^\delta x + \delta_\alpha^\delta \delta_\gamma^\beta (1-x)\right] [x(1-x)]^{-3/4} \tag{4.5b}$$

and an internal part which depends on the gauge bosons of interest. Equations (4.5) are easily obtained by bosonizing the spin-fields along the lines of [57], and show that we already get the correct kinematical structure,

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\bar\alpha\bar\beta\bar\gamma\bar\delta}(\Sigma_{\alpha\bar\alpha}\cdot\zeta)(\Sigma_{\beta\bar\beta}\cdot\zeta)(\Sigma_{\gamma\bar\gamma}\cdot\zeta)(\Sigma_{\delta\bar\delta}\cdot\zeta) = t_8 \zeta^4 \ . \tag{4.6}$$

4.1.2 Four-twist-field amplitude

We now consider more specifically the amplitude between four gauge bosons from the twisted sector. The internal part is given by the correlator of four twist fields on $T^4/\mathbb{Z}_2$. Since twist fields create a $\mathbb{Z}_2$ cut in the world-sheet, this is equivalent to a vacuum amplitude on the covering surface of the sphere with 4 punctures, namely a genus one surface [58]. This equivalence will turn out to be crucial for the heterotic-type II duality to hold. More precisely, the modular parameter of the torus is related to the vertex positions through the Picard map

$$x = \left(\frac{\vartheta_3}{\vartheta_4}\right)^4 \ , \quad \frac{dx}{x} = i\pi\vartheta_2^4 d\tau \ , \tag{4.7}$$

so that the four-point amplitude for $T^4/\mathbb{Z}_2$ twist fields is given by a slight adaptation from [29],

$$\langle H_{\epsilon_1}(\infty) H_{\epsilon_2}(1) H_{\epsilon_3}(x) H_{\epsilon_4}(0)\rangle = 2^{-8/3} \frac{Z_{4,4}\begin{bmatrix}\epsilon_i^1+\epsilon_i^4\\ \epsilon_i^1+\epsilon_i^2\end{bmatrix}}{\tau_2^2 \eta^4 \bar\eta^4} |x(1-x)|^{-1/3} \tag{4.8}$$

if charge conservation $\epsilon_i^1 + \epsilon_i^2 + \epsilon_i^3 + \epsilon_i^4 = 0 \mod 2$ is obeyed for every $i = 1 \dots 4$, zero otherwise. This selection rule results from a discrete group of symmetries of the orbifold CFT [48], which correspond to half lattice translations on the covering torus T^4, as well as their T-dual counterparts. The four translations exchange the 16 twisted sectors in pairs, while the T-dual translations act by -1 on eight of the 16 twisted sectors. These symmetries commute up to a global -1 factor on all twisted sectors, and thus generate a dihedral group $\mathbb{Z}_2 \ltimes \mathbb{Z}_2^8$ which generalizes the D_4

symmetry of the $S_1/\mathbb{Z}_2$ orbifold CFT. The above selection rule is precisely the one encountered on the heterotic side (3.18), providing new support for the duality.

Putting (4.8) together with (4.4) and (4.5), and changing the integration variable from x to τ, we therefore obtain

$$\Delta = \frac{l_{\rm II}^6}{V_{K_3}} \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} Z_{4,4} \left[{\epsilon_i^1+\epsilon_i^4 \atop \epsilon_i^1+\epsilon_i^2} \right] (G/l_{\rm II}^2, B) \ , \tag{4.9}$$

where we dropped an overall constant. The integration runs over the fundamental domain of the index 6 subgroup of $Sl(2,\mathbb{Z})$, which is the moduli space of the sphere with 4 punctures (see appendix A.2 for a discussion of congruence 2 subgroups of $Sl(2,\mathbb{Z})$). Note in particular, that the oscillators in (4.8) have dropped, in agreement with the fact that this amplitude should be half-BPS saturated. The normalization factor $l_{\rm II}^6/V_{K_3}$ has been chosen so as to agree with the heterotic result.

It is useful to discuss more specifically which shifts occur in the lattice partition function. Firstly, we note that a permutation of the four twist fields can be reabsorbed by a modular transformation which maps the extended fundamental domain $\mathcal{F}_2$ to itself, and hence leaves the integral invariant. We thus have only three possible results, up to permutations of the torus directions:
(i) if all twist fields sit at the same point,

$$\Delta_{(F_I)^4} = \frac{l_{\rm II}^6}{V_{K_3}} \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} Z_{4,4} \left[{0000 \atop 0000} \right] = 6 \frac{l_{\rm II}^6}{V_{K_3}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4} \left[{0000 \atop 0000} \right] \ , \tag{4.10}$$

(ii) if they are separated in two pairs,

$$\Delta_{(F_I)^2(F_J)^2} = \frac{l_{\rm II}^6}{V_{K_3}} \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} Z_{4,4} \left[{0000 \atop 0001} \right] = 3 \frac{l_{\rm II}^6}{V_{K_3}} \int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2} Z_{4,4} \left[{0000 \atop 0001} \right] \ , \tag{4.11}$$

(iii) if they sit at different fixed points, yet satisfying the selection rule,

$$\Delta_{F_I F_J F_K F_L} = \frac{l_{\rm II}^6}{V_{K_3}} \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^2} Z_{4,4} \left[{0100 \atop 1000} \right] \ . \tag{4.12}$$

Again, the precise shifts appearing in (4.11),(4.12) depend on the choice of twist fields, but the orbit structure is general. In the above amplitudes, we have implicitly subtracted the infrared divergence coming from the vacuum in the lattice partition functions, which from the point of view of the tree-level amplitude correspond to the exchange of massless particles.

4.1.3 Four-untwisted-field amplitude

In contrast to the previous case, the correlator between four untwisted fields does not involve the covering torus, and we have to deal with a genuine tree-level computation. The computation of various scattering amplitudes of four gauge bosons is then

identical to the analogous computation in the maximally supersymmetric type II theory. This computation has not been done to our knowledge but a quick argument already indicates that the four derivative couplings of (4,4) gauge bosons vanish at tree level. Indeed, the leading corrections to gravitational couplings occur at the 8-derivative level [59]. By supersymmetry, we expect that non-trivial corrections to Ramond-Ramond self-couplings should start at the eight-derivative level as well. Since Ramond-Ramond fields in ten dimensions descend to the (4,4) gauge fields upon compactification to 6 dimensions, it is evident that there should be no (four-derivative) F^4 terms for these fields. This of course does not preclude the existence of F^4 couplings mixing twisted and untwisted gauge fields.

The correlators of $SO(4)$ spin fields can be simply obtained by the usual bosonization techniques, and read

$$\langle \Psi_\alpha(\infty)\Psi_\beta(1)\Psi_\gamma(x)\Psi_\delta(0)\rangle = \frac{1}{\sqrt{x(1-x)}}\left[\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} - x\ \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\right] \ , \tag{4.13a}$$

$$\langle \Psi_\alpha(\infty)\Psi_{\dot\beta}(1)\Psi_\gamma(x)\Psi_{\dot\delta}(0)\rangle = \epsilon_{\alpha\gamma}\epsilon_{\dot\beta\dot\delta} \ , \tag{4.13b}$$

and similarly for the right-moving fields. Including the contribution from the ghost, 6D spin fields, as well as the momentum dependence $|x|^{-s/4}\,|1-x|^{-t/4}$ with $k_2\cdot k_3 = -t/2$, $k_3\cdot k_4 = -s/2$, $s+t+u=0$, the total amplitude reduces to a combination of standard integrals

$$\int d^2x |x|^{-a-s/4}|1-x|^{-b-t/4} = \pi\frac{\Gamma\left(1-\frac{a}{2}-\frac{s}{8}\right)\Gamma\left(1-\frac{b}{2}-\frac{t}{8}\right)\Gamma\left(\frac{a+b}{2}-1-\frac{u}{8}\right)}{\Gamma\left(2-\frac{a+b}{2}+\frac{u}{8}\right)\Gamma\left(\frac{a}{2}+\frac{s}{8}\right)\Gamma\left(\frac{b}{2}+\frac{t}{8}\right)} \tag{4.14}$$

with $(a,b)=(0,2),(2,0)$ and $(2,2)$ in the s,t,u-channels respectively. Expanding a typical contribution for small momenta, we have

$$A(s,t) = \frac{\Gamma(1-s/8)\Gamma(-t/8)\Gamma(-u/8)}{\Gamma(1+u/8)\Gamma(s/8)\Gamma(1+t/8)} = \frac{s^2}{8}\left(\frac{1}{stu}+\frac{\zeta(3)}{256}+\dots\right) \ . \tag{4.15}$$

The pole term corresponds to the tree-level massless exchange, and has to be subtracted in order to extract a correction to the effective action as in the twisted case. The correction only occurs at order s^2, corresponding to an eight-derivative coupling in the effective action. Hence there are no F^4 couplings at tree-level between four untwisted chiral fields, nor between four antichiral fields, as we anticipated at the beginning of this section. The first non-trivial correction however implies $\partial^4 F^4$ couplings, which are nothing but the ten-dimensional eight-derivative couplings, related by supersymmetry to the $t_8 t_8 R^4$ couplings [59], reduced on the torus T^4.

There is a puzzle concerning the $(4,0)^2(0,16)^2$ threshold. In the heterotic string this was shown to vanish. A type II computation along the lines above seems to give a non-zero answer. Clearly this deserves further study.

4.2 Duality and triality

Let us summarize the salient features of our computations so far, concentrating on the simple case of four-(0,16) gauge boson scattering for now.

- On the heterotic side, the one-loop amplitude was expressed as the integral over the fundamental domain of the lattice partition function of the torus T^4 with particular shifts, with an insertion of an elliptic genus $\Phi(\tau) = (\alpha E_4 + \beta \hat{E}_2^2)\vartheta_3^8\vartheta_4^8/\eta^{24}$. This structure is characteristic of half-BPS heterotic couplings, where the fermionic zero-modes are just saturated on the left-hand side and the right-moving oscillators generate the Hagedorn density of BPS states. Thanks to the Hecke identities described in Appendix B.3, the elliptic genus has dropped, leaving a simple integral of a shifted lattice partition function such as (3.10),(3.16) and (3.20). A particular selection rule (3.18) was also found.

- On the type IIA side, the tree-level amplitude of four twist fields has turned out to secretly be a genus 1 amplitude (4.9) on the covering of the 4-punctured sphere. The BPS nature of the coupling was revealed in the cancellation of the bosonic and fermionic determinants on the covering surface. The selection rule was a simple consequence of the $\mathbb{Z}_2 \ltimes \mathbb{Z}_2^8$ discrete symmetries of the orbifold. Eventually, the amplitude reduced to an integral (4.9) of the shifted partition function of the torus covering the K_3 surface, in agreement with the heterotic results (3.10), (3.16), (3.20). The tree-level amplitude for untwisted fields on the other hand was shown in (4.15) to vanish at 4 derivative order for $(0,d)$ or $(d,0)$ gauge bosons, in accordance with the heterotic result (3.27)[5].

However, it takes yet another miracle to identify the heterotic result with the type II result: indeed the two tori of moduli (g,b) and (G,B) are not identical, but, as we argued in Section 2.4, related by triality,

$$V_{K_3} = R_1^2\ , \quad \gamma = g_3\ , \quad \beta = B_{33}\ , \quad B_3 = A\ , \quad B_{\bar{3}} = B_{13}\ . \tag{4.16}$$

This transformation is certainly not a symmetry of the integrand $Z_{4,4}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, as a simple study of various decompactification limits makes clear. However, it has been shown in [10, 60] that the integrated result could be rewritten as an Eisenstein series in either the vector, spinor or the conjugate spinor representation,

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = \frac{1}{\pi}\, \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{V};s=1} = \frac{1}{\pi}\, \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{S};s=1} = \frac{1}{\pi}\, \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{C};s=1}\ , \tag{4.17}$$

which implies the identity of the heterotic and type II results,

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](g,b) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](G,B)\ . \tag{4.18}$$

[5] As mentioned before the issue of matching the $(0,4)^2(0,16)^2$ threshold remains obscure.

This claim was supported in [10] by showing that either of the terms in (4.17) was an eigenmode of the Laplacian operator on $[SO(4)\times SO(4)]\backslash SO(4,4,\mathbb{R})$, and of another non-invariant second order differential operator as well; it was also shown that the large volume and decompactification limits agreed. The same arguments can also be made for the two terms of (4.18) without using Eisenstein series as an intermediate step. While heterotic-type II duality clearly implies (4.18), it would be useful to have a mathematically rigorous proof for it.

In the case of four identical gauge fields, the identity (4.17) directly matches the heterotic result (3.10) with the corresponding type II result (4.10). More generally, we need an extension of this identity to the case with half-integer shifts. This can be obtained by re-expressing the shifted lattice sums as unshifted lattice sums with redefined moduli, and apply the triality (4.17). For example, we may rewrite the heterotic amplitude

$$\int_{\mathcal{F}_2^-} Z_{4,4}\begin{bmatrix}0\\ \mathrm{d}\end{bmatrix}(g,b) = \frac{12}{\pi}\,\mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{V};s=1}(g/2,b/2) = \frac{12}{\pi}\sum_{m_i\in 2\mathbb{Z},n^i\in\mathbb{Z}}\frac{2}{\mathcal{M}^2_{\boldsymbol{V}}(g,b)} \tag{4.19a}$$

$$= \frac{12}{\pi}\sum_{m,m_{ij}\in 2\mathbb{Z},m^{1j},n\in\mathbb{Z}}\frac{2}{\mathcal{M}^2_{\boldsymbol{C}}(G,B)} = \frac{12}{\pi}\,\mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{C};s=1}(R_1/2) \tag{4.19b}$$

$$= \int_{\mathcal{F}_2^-}\left(6Z_{4,4}\begin{bmatrix}0000\\1000\end{bmatrix}(G,B) + 2Z_{4,4}\begin{bmatrix}0000\\0000\end{bmatrix}(G,B)\right) \tag{4.19c}$$

in a form suitable for comparison with type II amplitudes. Here, we have used (A.20), (A.21d) in the first step to convert to an Eisenstein series in the vector representation, in the second step we have rewritten this series as a constrained Eisenstein series involving the vector mass at the original heterotic moduli. The third step consists of the application of the triality map (2.15), (2.17) to write the vector mass as a conjugate spinor mass with type II moduli, which is re-expressed as a conjugate spinor Eisenstein series in the fourth step. Then, the fifth step uses again (A.21d), (A.20) to present the result as a tree-level type II amplitude of the form (4.9). However, the precise matching will require the exact identification of the gauge fields on the type II side with those on the heterotic side, which we have not been able to achieve. It would be also interesting to understand how the duality holds at other orbifold points of K_3, since naively the correlator of $\mathbb{Z}_n$ twist fields on the sphere involves higher genus Riemann surfaces, albeit of a very symmetric type.

Finally, let us comment on R^4 gravitational couplings. In that case, the one-loop heterotic result translates into a two-loop contribution on the type IIA side. On the other hand, it is known that there is a $t_8t_8R^4$ coupling arising at tree-level and one-loop on the type IIA side, which translate into a two- and three-loop contribution on the heterotic side. It would be interesting to carefully determine the combination of these gravitational couplings that obeys a non-renormalization theorem, if any. It would also be very interesting to compute higher-derivative R^4F^{4g-4} couplings on

the heterotic side, and compare them with the topological amplitudes on the type II side [52].

5. Dual interpretation of higher derivative couplings

Having reproduced the type IIA tree-level F^4 coupling in 6 dimensions from the heterotic one-loop amplitude, we now would like to use the duality map to obtain some non-trivial results in other dimensions. This will provide further checks of duality, and at the same time give new insights into non-perturbative effects on the dual side.

5.1 Type I' thresholds

As discussed in Section 2.2, the heterotic string on S_1 at the $SO(16)\times SO(16)$ point is dual to type I' with eight D8-branes located at each of the two orientifold points. The modular integrals of shifted lattices are quite simple to compute using the summation identity (A.12) together with the modular integral (A.21a) for an unshifted lattice. For $(0,16)$ gauge fields, we then obtain from (3.12), (3.11) and (3.17),

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}}^4} = \frac{1}{6}\left(2I_1(R_{\mathrm{H}}) - I_1(R_{\mathrm{H}}/2)\right) = \frac{\pi}{3}\frac{R_{\mathrm{H}}}{l_{\mathrm{H}}^2}\ , \tag{5.1a}$$

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}}^2\mathrm{Tr}F_{\mathrm{d}}^2} = -\Delta_{\mathrm{Tr}F_{\mathrm{d}_1}^2\,\mathrm{Tr}F_{\mathrm{d}_2}^2} = -\frac{1}{6}\left(I_1(R_{\mathrm{H}}) - 2I_1(R_{\mathrm{H}}/2)\right) = \frac{\pi}{3R_{\mathrm{H}}}\ , \tag{5.1b}$$

where we reinstated the powers l_{H} on dimensional ground. Translating to type I' variables using (2.7), the heterotic thresholds translate into

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}}^4} = \frac{\pi}{3g_{\mathrm{I'}}l_{\mathrm{I'}}}\ ,\quad \Delta_{\mathrm{Tr}F_{\mathrm{d}}^2\mathrm{Tr}F_{\mathrm{d}}^2} = -\Delta_{\mathrm{Tr}F_{\mathrm{d}_1}^2\,\mathrm{Tr}F_{\mathrm{d}_2}^2} = \frac{\pi R_{\mathrm{I'}}}{3l_{\mathrm{I'}}^2}\ . \tag{5.2}$$

Given the heterotic non-renormalization theorem, these couplings should therefore be given by a disk and cylinder diagram respectively, without further corrections. In particular, note that the absence of factorized couplings $(\mathrm{Tr}F^2)^2$ at tree-level is consistent with the fact that these couplings need (at least) two boundaries. Moreover, the absence of non-perturbative corrections is consistent with the fact that there are no half-BPS instantons in type I' in 9 dimensions. The F^4 couplings have been studied in [14] in the context of the duality between the $SO(32)$ heterotic string and type I, where it was noticed that the duality requires contributions of higher genus surfaces ($\chi=-1,-2$) on the type I side, due to non-holomorphic contributions to the elliptic genus. The $SO(16)\times SO(16)$ point therefore appears to be a simpler setting to further understand heterotic-type I duality, and this is indeed the point where this duality can be derived from the eleven-dimensional strong coupling dynamics of the heterotic string [40].

We may also consider how the gravitational couplings (3.23) translate under the duality. In that case,

$$\Delta_{\mathrm{Tr}R^4} = 4\Delta_{(\mathrm{Tr}R^2)^2} = \frac{\pi}{48}\left(\frac{R_{\mathrm{H}}}{l_{\mathrm{H}}^2} + \frac{2}{R_{\mathrm{H}}}\right) = \frac{\pi}{48}\left(\frac{1}{g_{\mathrm{I}'} l_{\mathrm{I}'}} + 2\frac{R_{\mathrm{I}'}}{l_{\mathrm{I}'}^2}\right) , \tag{5.3}$$

so that they receive contributions both at tree-level and one-loop on the type I' side. As already discussed, it is unclear if the non-renormalization theorem applies on the heterotic side, and they may therefore get contributions from higher loops.

5.2 F-theory on K_3 and O7-plane interactions

As discussed in Section 2.3, the heterotic string on T^2 at the $SO(8)^4$ point is dual to type IIB on a $T^2/\mathbb{Z}_2$ orientifold, which is nothing but F-theory on K_3 at the orbifold point. The F^4 and related couplings have been considered in detail in [23] and it is a useful check on our formalism[6] to rederive their results. For $d = 2$ (as for $d = 1$) the modular integrals can be evaluated thanks to the summation identities of Appendix A.2 and the explicit modular integral (A.21b) for the unshifted lattice. For the $(0, 16)$ gauge couplings (3.12), (3.11) in a given $SO(8)$, we can use the identity (A.20) along with the explicit result (A.21b) to obtain

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}}^4} = I_2(T, U) - I_2(T/2, U) = -\log\frac{2|\eta(T)|^4}{|\eta(T/2)|^4} , \tag{5.4a}$$

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}}^2 \mathrm{Tr}F_{\mathrm{d}}^2} = I_2(T/2, U) - \frac{1}{2} I_2(T, U) = -\frac{1}{2}\log\left[\frac{2\pi e^{1-\gamma_E}}{3\sqrt{3}}\frac{T_2 U_2 |\eta(T/2)|^8 |\eta(U)|^4}{|\eta(T)|^4}\right] . \tag{5.4b}$$

Under the duality, T, U map to the complex structures U_{F}, U_{B} of the fiber and the base respectively, so that (5.4a), (5.4b) appear to give a tree-level result together with an infinite series of D-instanton corrections of classical action $S_{\mathrm{cl}} = N U_{\mathrm{F}}$. Such effects have been discussed in a related context in [21].

For the couplings (3.16) between the four different $SO(8)$ factors we have 3 pairs of possibilities, for which we use the summation identities (A.13a), (A.13b), (A.13c) respectively, yielding after some algebra

$$\Delta_{01} = \Delta_{23} = I_2(T/2, U/2) - I_2(T/2, U) = \log\frac{2|\eta(U)|^4}{|\eta(U/2)|^4} , \tag{5.5a}$$

$$\Delta_{02} = \Delta_{13} = I_2(T/2, 2U) - I_2(T/2, U) = \log\frac{|\eta(U)|^4}{2|\eta(2U)|^4} , \tag{5.5b}$$

$$\Delta_{03} = \Delta_{12} = I_2(T/2, (U+1)/2) - I_2(T/2, U) = \log\frac{2|\eta(U)|^4}{|\eta(\frac{U+1}{2})|^4} . \tag{5.5c}$$

[6] and on the results of [23] as well, some of which have been corrected in the erratum in [23].

In that case, the F-theory couplings arise at one-loop only from the point of view of the IIB orientifold perturbative description. This is consistent with the fact that gauge bosons from different branes have to be inserted on opposite sides of the cylinder in a one-loop computation. For the gravitational couplings (3.23), we find instead

$$\Delta_{\mathrm{Tr}R^4} = 4\Delta_{(\mathrm{Tr}R^2)^2} = \frac{1}{16} I_2(T/2, U) = -\frac{1}{16} \log \left[\frac{4\pi e^{1-\gamma_E}}{3\sqrt{3}} T_2 U_2 |\eta(T/2)|^4 |\eta(U)|^8 \right] , \tag{5.6}$$

which exhibits an infinite series of D-instanton effects from expanding the Dedekind function.

5.3 M theory on K_3 and enhanced gauge symmetry

We now turn to the heterotic string compactified on T^3 at the $SU(2)^{16}$ point. One dual description is provided by type IIA on a $T^3/\mathbb{Z}_2$ orientifold, which is similar to the two previous cases. We are however more interested in the M-theory description, which involves compactification on K_3 with A_1 conical singularities (see Section 2.5). Each of the 8 fixed points of the $T^3/\mathbb{Z}_2$ orientifold has thus split into two distinct fixed points of the $T^4/\mathbb{Z}_2$ M-theory orbifold. Using the duality map (2.18), we find that the F^4 couplings (3.12), (3.11), (3.16) translate into

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}}^4} = \frac{l_{\mathrm{M}}^3}{12\sqrt{V_{K_3}}} \int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2} \left(2Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] - Z\left[\begin{smallmatrix}0\\\mathrm{d}\end{smallmatrix}\right] \right)(\gamma,\beta) , \tag{5.7a}$$

$$\Delta_{(\mathrm{Tr}F_{\mathrm{d}}^2)^2} = \frac{l_{\mathrm{M}}^3}{12\sqrt{V_{K_3}}} \int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2} Z\left[\begin{smallmatrix}0\\\hat{\mathrm{d}}\end{smallmatrix}\right](\gamma,\beta) , \tag{5.7b}$$

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}_1}^2 \mathrm{Tr}F_{\mathrm{d}_2}^2} = \frac{l_{\mathrm{M}}^3}{12\sqrt{V_{K_3}}} \int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2} \left(2Z\left[\begin{smallmatrix}0\\\hat{\mathrm{d}}\end{smallmatrix}\right] - 3Z\left[\begin{smallmatrix}00\\\mathrm{d}1\end{smallmatrix}\right] \right)(\gamma,\beta) , \tag{5.7c}$$

where (γ,β) encode the shape of the orbifold $T^4/\mathbb{Z}_2$. The 3-digit numbers d label one of the 8 copies of the gauge group $SO(4) = SU(2)\times SU(2)$. We can rewrite these results in a more appealing fashion by using the representation (A.21c) of the modular integrals in terms of $SO(3,3,\mathbb{Z})$ Eisenstein series in the spinor representation along with the identity (A.20). For example, the $\mathrm{Tr}(F_{\mathrm{d}})^4$ coupling (5.7a) can be rewritten as

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}}^4} = \frac{l_{\mathrm{M}}^3}{2\pi\sqrt{V_{K_3}}} \left[\mathcal{E}^{SO(3,3,\mathbb{Z})}_{\boldsymbol{S};s=1}(\gamma,\beta) - \sqrt{2}\, \mathcal{E}^{SO(3,3,\mathbb{Z})}_{\boldsymbol{S};s=1}(\gamma/2,\beta/2) \right] , \tag{5.8}$$

where the $SO(3,3,\mathbb{Z})$ Eisenstein series is defined by [10]:

$$\mathcal{E}^{SO(3,3,\mathbb{Z})}_{\boldsymbol{S};s=1}(\gamma,\beta) = \hat{\sum_{m^{1,2,3},n\in\mathbb{Z}}} \frac{\sqrt{\det\gamma}}{(m^i+\beta^i n)\gamma_{ij}(m^j+\beta^j n) + (\det\gamma) n^2} \tag{5.9}$$

with $\beta^i = \epsilon^{ijk}\beta_{jk}/2$, and the hat restricts the sum to non-zero integers. Using (2.13), we can re-express this in terms of the metric G of the type IIA orbifold

$$\mathcal{E}^{SO(3,3,\mathbb{Z})}_{\boldsymbol{S};s=1}(\gamma,\beta) = \hat{\sum_{m^r\in\mathbb{Z}}} \frac{\sqrt{V_{K_3}}}{m^r G_{rs} m^s} = \sqrt{V_{K_3}}\ \mathcal{E}^{Sl(4,\mathbb{Z})}_{\mathbf{4};s=1}(G)\ , \tag{5.10}$$

where m_r can be thought of as momenta along T^4. We can now rewrite (5.8) in terms of Eisenstein series for a congruence 2 subgroup of $Sl(4,\mathbb{Z})$,

$$\Delta_{\mathrm{Tr}F_{\mathrm{d}}^4} = \frac{l_{\mathrm{M}}^3}{2\pi}\left[\hat{\sum_{m^r\in\mathbb{Z}}} \frac{1}{(m^r G_{rs} m^s)} - 4 \hat{\sum_{\substack{m^{2,3,4}\in 2\mathbb{Z}\\ m^1\in\mathbb{Z}}}} \frac{1}{(m^r G_{rs} m^s)}\right] . \tag{5.11}$$

The fact that the direction 1 is singled out should not come as a surprise, since the 16 orbifold fixed points originate from the 8 orientifold points which have split along direction 1. The 16 fixed points should however appear on the same footing from the M-theory point of view. This is indeed so, since, upon decomposing the $SO(4)$ gauge field $F_{\mathrm{d}} = F_{\mathrm{d0}} \otimes 1 + 1 \otimes F_{\mathrm{d1}}$ into its $SU(2)\times SU(2)$ components and using the identity $\mathrm{Tr}F^4 = (\mathrm{Tr}F^2)^2$ for $SU(2)$ gauge fields, we have

$$\mathrm{Tr}F^4 = \mathrm{Tr}F_{\mathrm{d0}}^4 + \mathrm{Tr}F_{\mathrm{d1}}^4 + 6\mathrm{Tr}F_{\mathrm{d0}}^2\mathrm{Tr}F_{\mathrm{d1}}^2\ , \tag{5.12a}$$

$$(\mathrm{Tr}F^2)^2 = \mathrm{Tr}F_{\mathrm{d0}}^4 + \mathrm{Tr}F_{\mathrm{d1}}^4 + 2\mathrm{Tr}F_{\mathrm{d0}}^2\mathrm{Tr}F_{\mathrm{d1}}^2\ . \tag{5.12b}$$

The $SO(4)$ gauge couplings (5.11) can thus be rewritten as $SU(2)$ gauge couplings

$$\Delta_{\mathrm{Tr}F_{\mathrm{d0}}^4} = \Delta_{\mathrm{Tr}F_{\mathrm{d1}}^4} = \frac{l_{\mathrm{M}}^3}{4\pi} \hat{\sum_{m^r\in\mathbb{Z}}} \frac{1}{(m^r G_{rs} m^s)}\ , \tag{5.13a}$$

$$\Delta_{\mathrm{Tr}F_{\mathrm{d0}}^2\mathrm{Tr}F_{\mathrm{d1}}^2} = \frac{l_{\mathrm{M}}^3}{2\pi}\left[5\hat{\sum_{m^r\in\mathbb{Z}}} \frac{1}{(m^r G_{rs} m^s)} - 16 \hat{\sum_{\substack{m^{2,3,4}\in 2\mathbb{Z}\\ m^1\in\mathbb{Z}}}} \frac{1}{(m^r G_{rs} m^s)}\right] . \tag{5.13b}$$

The $\mathrm{Tr}F_{\mathrm{d0}}^4$ now makes no reference to any particular direction as it should, while the $\mathrm{Tr}F_{\mathrm{d0}}^2\mathrm{Tr}F_{\mathrm{d1}}^2$ singles out the direction 1 along which the two gauge fields are separated.

The results (5.13) are given exactly at first order in $l_{\mathrm{M}}^2/\sqrt{V_{K_3}}$, which is the natural expansion parameter on the M-theory side problem. They cannot however be obtained from eleven-dimensional supergravity in perturbation theory due to the conical singularity, and it is necessary to include the M2-brane in order to provide the $SU(2)$ degrees of freedom. It would be very interesting to devise a perturbative approach in this situation, perhaps along the lines of [61], in order to recover the

result (5.11). We also note that the F^4 couplings that we have computed in M-theory on $T^4/\mathbb{Z}_2$ also give the F^4 couplings in type IIA on $T^4/\mathbb{Z}_2$ in the absence of B-flux on the vanishing cycles, where the conformal field theory is singular. Surprisingly, they are finite. They should presumably correspond to the finite part of the F^4 amplitude when the singularity has been subtracted, and it would be interesting to analyze the behaviour of the amplitude when the B-flux is perturbed away from zero.

5.4 Type IIA on $K_3 \times S_1$, IIB on K_3 and D-instantons

For $d > 4$, the dual description of the heterotic string compactified on T^d at the special $U(1)^{16}$ point now allows for non-perturbative effects. In particular, for $d = 5$, the type IIA string theory compactified on $K_3 \times S_1$ has instanton configurations coming from even D-branes whose Euclidean world-volume is wrapped on even-cycles of K_3 times the circle S_1. In the type IIB picture, instanton configurations exist already in 6 dimensions, as odd Euclidean D-branes wrapped on even cycles of K_3. These effects were first computed in [26], and here we want to get a more quantitative understanding of them.

From the duality relation $R_{\rm H}/l_{\rm H} = R_{\rm A}/(g_{6\rm IIA} l_{\rm II}) = 1/g_{6\rm IIB}$, we see that the weak coupling regime on the type IIA or IIB side corresponds to the decompactification limit $R_{\rm H} \gg l_{\rm H}$ on the heterotic side. In this limit, the heterotic result exhibits a series of world-sheet instanton contributions which will be interpreted as D-instanton effects on the type II side. For simplicity, we will focus on the $\mathrm{Tr}(F_{\rm d})^4$ couplings in (3.10), given by a modular integral of an unshifted partition function,

$$\Delta_{5D} = l_{\rm H}^3 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{5,5}(g/l_{\rm H}^2, b, R_{\rm H}/l_{\rm H}, w) \ , \tag{5.14}$$

where we dropped the numerical factor, and denoted by w the Wilson lines of the six-dimensional $(4,4)$ gauge fields around the extra circle. We will comment on the effects of shifts at the end.

In order to determine the large $R_{\rm H}$ behaviour of (5.14), it is convenient to adopt a Lagrangian representation for the S^1 part and a Hamiltonian representation for the T^4 part:

$$\Delta_{5D} = l_{\rm H}^2 R_{\rm H} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{p,q} \sum_{m_i, n^i} \exp\left(-\pi \frac{R_{\rm H}^2 |p - \tau q|^2}{l_{\rm H}^2 \tau_2} + 2\pi i p \ w_i n^i \right) \tau_2^2 q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} \ , \tag{5.15}$$

where m_i, n^i denote the momenta and windings on T^4. We apply the standard orbit decomposition method on the integers (p,q), trading the sum over $Sl(2,\mathbb{Z})$ images of (p,q) for a sum over images of the fundamental domain $\mathcal{F}$ [62] (see [18, 10] for relevant formulae). The zero orbit gives back the six-dimensional result (4.18) up to a volume factor, and reproduces the tree-level type II contribution in 5 dimensions:

$$\Delta_{5D}^{\rm zero} = l_{\rm H}^2 R_{\rm H} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4}(g/l_{\rm H}^2, b) = R_{\rm A} \Delta_{\rm IIA}^{\rm tree} \ . \tag{5.16}$$

The degenerate orbit on the other hand, with representatives $(p, 0)$, can be unfolded onto the strip $|\tau_1| < 1/2$. The τ_1 integral then imposes the level matching condition $p_L^2 - p_R^2 = 2m_i n^i = 0$, and the τ_2 integral can be carried out in terms of Bessel functions to give

$$\Delta_{5D}^{\rm deg} = 2 l_{\rm H} R_{\rm H}^2 \sum_{p\neq 0} \sum_{(m_i,n^i)\neq 0} \delta(m_i n^i) \frac{|p|}{\sqrt{m^t M_{4,4} m}} K_1 \left(2\pi \frac{R_{\rm H}}{l_{\rm H}} |p| \sqrt{m^t M_{4,4} m} \right) e^{2\pi i p w_i n^i} \ , \tag{5.17}$$

up to a divergent contribution $\pi^2 R_{\rm H}^3 \Gamma(-1)/3$, coming from the origin of the $(4,4)$ lattice, which we assume to be regularized. Here $m^t M_{4,4} m = p_L^2 + p_R^2$ with $m = (m^i, n_i)$. It is straightforward to translate this result to the type IIA side,

$$\Delta_{5D}^{\rm deg} = 2 g_{6\rm IIA} l_{\rm II} R_{\rm A}^2 \sum_{p\neq 0} \sum_{(m_i,n^i)\neq 0} \delta(m_i n^i) \tag{5.18}$$
$$\cdot \frac{|p|}{\sqrt{m^t M_{4,4} m}} K_1 \left(2\pi \frac{R_{\rm A}}{g_{6\rm IIA} l_{\rm II}} |p| \sqrt{m^t M_{4,4} m} \right) e^{2\pi i p w_i n^i} \ ,$$

where $M_{4,4}$ is now the mass matrix (2.12) of D-brane states wrapped on the untwisted cycles of $T^4/\mathbb{Z}_2$. Given the asymptotic behaviour $K_1(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$, we see that this is a sum of order e^{-1/g_s} non-perturbative effects corresponding to $N = pr$ Euclidean (anti) D-branes wrapped on S_1 times a cycle of homology charges $(m_i, n^i)/r$ on T^4, where r is the greatest common divisor of (m_i, n^i).

It is worth pointing out a number of peculiarities of the result (5.18). First, due to the absence of a holomorphic insertion in (5.14), all instanton effects are due to *untwisted* D-branes wrapped along even cycles of K_3, even though we are discussing F^4 couplings between fields located on the fixed points of the orbifold. This is in contrast to the result in four-derivative scalar couplings [26], where a contribution from the whole Hagedorn density of BPS states was found. This is an important simplification due to our choice of the orbifold point in the K_3 moduli space. Second, the integration measure corresponding to a given number of D-branes N is easily seen to be $\sum_{r|N}(1/r^2)$, where r runs over the divisors of N, just as in the case of D-instanton effects in theories with 32 supersymmetries [51, 2]. This is an unexpected result, since the bulk contribution to the index for the quantum mechanics with 8 unbroken symmetries is $1/N^2$ instead [3], which did arise in four-derivative scalar couplings at the enhanced symmetry point [26, 63]. Finally, it is clear that the above analysis goes through in the case with shifts on the lattice, since those only affect the momenta and windings on T^4. They translate into corresponding shifts on the lattice of D-instantons contributing to ${\rm Tr}F^4$.

The situation from the T-dual type IIB point of view is also interesting. From the mapping (2.22), we see that the one-loop heterotic F^4 coupling in 5 dimensions

translates into

$$\Delta_{5D} = \left(\frac{l_{\rm P}^2}{R_{\rm B}}\right)^3 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{5,5} \ , \tag{5.19}$$

where now $Z_{5,5}$ depends on the K_3 untwisted moduli and on the six-dimensional string coupling, but not on the size of the circle S^1 in six-dimensional Planck units. We can therefore simply take the limit $R_{\rm B} \to \infty$ to recover a six-dimensional amplitude. The powers of $R_{\rm B}$ in (5.19) are precisely such as to yield a finite $t_{12}H^4$ coupling in 6 dimensions, where H is one of the 16 anti-self-dual three-form field strengths arising from the twisted sectors of type IIB compactified on $T^4/\mathbb{Z}_2$, and t_{12} is a 12-index tensor constructed from t_8. We therefore get

$$\Delta_{H^4}^{\rm IIB} = l_{\rm P}^6 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{5,5} \tag{5.20}$$

which is the exact non-perturbative coupling of four self-dual twisted three-forms, invariant under the $SO(5,5,\mathbb{Z})$ subgroup of the U-duality group $SO(5,21,\mathbb{Z})$ left unbroken by the choice of the external legs. The above analysis of the heterotic decompactification limit still holds, and yields the tree-level and D-instanton contributions to this amplitude,

$$\begin{aligned}\Delta_{H^4}^{\rm IIB} =& \frac{g_{\rm II}^2 l_{\rm II}^{10}}{V_{K_3}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4} + 2\left(\frac{g_{\rm II}^2 l_{\rm II}^{10}}{V_{K_3}}\right)^{3/2} \cdot \\ &\cdot \sum_{p\neq 0} \hat{\sum_{m_i,n^i}} \delta(m_i n^i) \frac{\sqrt{m^t M_{4,4} m}}{|p|} K_1\left(2\pi \frac{V_{K_3}^{1/2}}{g_{\rm II} l_{\rm II}^2} |p| \sqrt{m^t M_{4,4} m}\right) e^{2\pi i p w_i n^i}\end{aligned} \tag{5.21}$$

which exhibits non-perturbative contributions from odd D-branes wrapped on even untwisted cycles of K_3. In particular, the ten-dimensional decompactification limit $V_{K_3} \gg l_{\rm II}^4$ reproduces the R^4 couplings in type IIB, as demonstrated in [26].

5.5 Type II on $K_3 \times T^2$ and NS5-brane corrections

Finally, we would like to discuss the four-dimensional case, which on the type II side receives corrections from NS5-branes wrapped on $K_3 \times T^2$. Similar corrections could also in principle arise on the heterotic side from 5-branes wrapped on T^6, but they do not affect four-gauge-boson couplings from the right-moving sector according to our conjecture. From the duality map $T_{\rm H} = S_{\rm IIA} = S_{\rm IIB}$, the weak coupling regime on the type II side again corresponds to the limit where the heterotic T^2 decompactifies.

The study of the decompactification limit proceeds as in (5.15) by performing an orbit decomposition on the integers running in the Lagrangian representation of the T^2 lattice, and the zero orbit and degenerate orbit reproduce the tree-level and D-instanton contributions on the type II side. The novelty in that case is that there is a third orbit, namely the non-degenerate orbit, which contributes as well. The

integral on τ_1 is Gaussian, and the subsequent integral along τ_2 is again given by a Bessel function. Before carrying out this integration, it is more enlightening to determine the saddle point, which controls the instanton effects at leading order. The saddle point equations are easily found to be

$$q^I g_{IJ}(p^J - \tau_1 q^J) + i\tau_2 m_i n^i = 0 \ , \qquad (5.22a)$$

$$-(p^I - \tau_1 q^I) g_{IJ}(p^J - \tau_1 q^J) + \tau_2^2 (q^I g_{IJ} q^J + m^t M_{4,4} m) = 0 \ , \qquad (5.22b)$$

where p^I and q^I are the integers running in the T^2 lattice partition function, and should be summed over $Sl(2,\mathbb{Z})$ orbits such that $p^1 q^2 - p^2 q^1 \neq 0$ only. g_{IJ} is the metric on T^2 in heterotic units. The solution of these equations is given by

$$\tau_1 = \frac{pq}{q^2} + i\frac{m_i n^i}{q^2}\sqrt{\frac{p^2q^2 - (pq)^2}{(q^2)^2 + q^2 m^t M_{4,4} m + (m_i n^i)^2}} \qquad (5.23a)$$

$$\tau_2 = \sqrt{\frac{p^2q^2 - (pq)^2}{(q^2)^2 + q^2 m^t M_{4,4} m + (m_i n^i)^2}} \ , \qquad (5.23b)$$

where contractions with g_{IJ} are understood, and corresponds to a classical action

$$S_{\rm cl} = 2\pi\sqrt{(p^2q^2 - (pq)^2)\left(1 + \frac{m^t M_{4,4} m}{q^2} + \frac{(m_i n^i)^2}{(q^2)^2}\right)} + 2\pi i \frac{(pq)(m_i n^i)}{q^2} + 2\pi i p B q \ . \qquad (5.24)$$

Reinstating the $l_{\rm H}$ dependence and mapping to dual type IIA variables using (2.9), the real part of the classical action

$$\Re S_{\rm cl} = 2\pi\sqrt{\frac{p^2q^2 - (pq)^2}{(q^2)^2}\left(\frac{(q^2)^2}{g_{\rm 6IIA}^4 l_{\rm II}^4} + \frac{q^2 m^t M_{4,4} m}{g_{\rm 6IIA}^2 l_{\rm II}^2} + (m_i n^i)^2\right)} \qquad (5.25)$$

scales as $1/g_{\rm 6IIA}^2$. The corresponding non-perturbative effects should therefore be interpreted as coming from $N = |p^1 q^2 - p^2 q^1|$ NS5-branes wrapped on $K_3 \times T^2$, and bound to D-brane states wrapped on an even cycle of K_3 times a circle on T^2 determined by the integers q^1, q^2. The result of the τ integration thus gives

$$\Delta_{4D}^{\rm n.d.} = 4 l_{\rm H}^4 \sum_{p^i, q^i} \sum_{m_i, n^i} \left(\frac{(q^2)^2 + q^2 m^t M_{4,4} m + (m_i n^i)^2}{p^2 q^2 - (pq)^2}\right)^{3/4} K_{3/2}\left(\Re S_{\rm cl}\right) e^{i\Im S_{\rm cl}} \ . \qquad (5.26)$$

In particular, we may look at the contribution of pure NS5-brane instantons, corresponding to $m_i = n^i = 0$. Choosing the orbit representatives as

$$\begin{pmatrix} q^1 & p^1 \\ q^2 & p^2 \end{pmatrix} = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix} \ , 0 \leq j < k \ , \quad p \neq 0 \ , \qquad (5.27)$$

and using the exact expression for the Bessel function

$$K_{3/2}(x) = \sqrt{\pi/2x}\,(1+1/x)\,e^{-x} \tag{5.28}$$

we obtain

$$\Delta_{4D}^{\rm NS5} = 2(g_{6\rm IIA} l_{\rm II})^4 U_2 \sum_N \mu(N) \left(N + \frac{1}{2\pi S_2} \right) e^{-2\pi N S_2} \left(e^{2\pi i N S_1} + e^{-2\pi i N S_1} \right) \ , \tag{5.29}$$

where we used the type II variable $S = a + iV_{K_3}V_{T^2}/(g_{\rm II}^2 l_{\rm II}^6)$ and extracted the instanton measure

$$\mu(N) = \sum_{r|N} \frac{1}{r^3}\ , \qquad \text{(NS5-brane on } K_3\times T^2) \ . \tag{5.30}$$

This result gives a prediction for the index (or rather the bulk contribution thereto) of the world-volume theory of the type II NS5-brane wrapped on $K_3 \times T^2$. It is a challenging problem to try and derive this result from first principles. It is also remarkable that, in virtue of (5.28) and in contrast to D-instantons, the NS5-instantons contributions do not seem to receive any perturbative subcorrections beyond one-loop.

It is interesting to compare this result to the corresponding index of the heterotic 5-brane wrapped on T^6, which can be extracted from the non-perturbative R^2 couplings in the heterotic string compactified on T^6 [17, 11]. Those can be computed by duality from the one-loop exact R^2 couplings in type II on $K_3 \times T^2$ [15], and read

$$\Delta_{R^2} = \hat{\mathcal{E}}^{Sl(2,\mathbb{Z})}_{\mathbf{2};s=1} = -\pi \log(S_2|\eta(S)|^4) \tag{5.31a}$$

$$= \frac{\pi^2}{3} S_2 + 2\pi\sqrt{S_2} \sum_N \mu(N) e^{-2\pi N S_2} \left(e^{2\pi i N S_1} + e^{-2\pi i N S_1} \right) \ . \tag{5.31b}$$

The summation measure turns out to be different from (5.30) and given instead by

$$\mu(N) = \sum_{r|N} \frac{1}{r}\ , \qquad \text{(Het 5-brane on } T^6) \ . \tag{5.32}$$

It is also worthwhile to notice that there are no subleading corrections around the instanton in the heterotic 5-brane case, whereas, by virtue of (5.28), these corrections occur at first order only in the type II NS5-brane on $K_3 \times T^2$. This is in contrast to D-instantons, for which the saddle point approximation to the Bessel function K_1 is not exact. It would be interesting to have a deeper understanding of these non-renormalization properties, possibly using the CFT description of the 5-brane [64].

Appendices

A. Shifted partition functions and lattice integrals

A.1 Hamiltonian and Lagrangian representation

As discussed in Section 3.1, the compactification on a torus with half-integer Wilson lines (1.3) is most conveniently described in terms of shifted lattice sums, which in the Hamiltonian representation read

$$Z_{d,d}\begin{bmatrix}h^i\\g^i\end{bmatrix}(g,b,\tau)=\tau_2^{d/2}\sum_{m_i,n^i\in\mathbb{Z}}(-)^{m_i g^i}q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}\ . \tag{A.1}$$

The left-moving and right moving momenta p_L, p_R are given by

$$p^i_{\substack{L\\R}}=n^i+\frac{h^i}{2}\pm g^{ij}\left[m_i+B_{ij}\left(n^j+\frac{h^j}{2}\right)\right]\ , \tag{A.2}$$

and the integers h^i, g_i are defined modulo 2, and when non-zero, break the T-duality $O(d,d,\mathbb{Z})$ to a finite index subgroup. Modular invariance on the other hand is manifest in the Lagrangian representation, obtained after Poisson resumming on the momenta m_i:

$$Z_{d,d}\begin{bmatrix}h^i\\g^i\end{bmatrix}(g,b,\tau)=V\sum_{\substack{m^i\in\mathbb{Z}+g^i/2\\n^i\in\mathbb{Z}+h^i/2}}\exp\left(-\frac{\pi}{\tau_2}\left(m^i-\tau n^i\right)g_{ij}\left(m^i-\bar{\tau}n^i\right)+2\pi i m^i B_{ij}n^j\right)\ . \tag{A.3}$$

In particular, insertions of left-moving and right-moving momenta in the Hamiltonian representation translate into

$$p^i_L\to-\frac{m^i+n^i\bar{\tau}}{i\tau_2}\ ,\quad p^i_R\to\frac{m^i+n^i\tau}{i\tau_2}\ , \tag{A.4}$$

where the m^i and n^i are integers shifted by $g^i/2$ and $h^i/2$ respectively. This translation is up to contractions which are easily fixed by demanding modular invariance. In particular, under modular transformations of τ, p_L and p_R have modular weight (1,0) and (0,1) respectively.

When h^i or g^i is non-zero, the shifted blocks (A.3) are not modular invariant. Instead, they transform among themselves as

$$T:\qquad Z_{d,d}\begin{bmatrix}h^i\\g^i\end{bmatrix}(\tau+1)=Z_{d,d}\begin{bmatrix}h^i\\g^i+h^i\end{bmatrix}(\tau)\ , \tag{A.5a}$$

$$S:\qquad Z_{d,d}\begin{bmatrix}h^i\\g^i\end{bmatrix}\left(-\frac{1}{\tau}\right)=Z_{d,d}\begin{bmatrix}g^i\\h^i\end{bmatrix}(\tau)\ , \tag{A.5b}$$

so that like T-duality, modular invariance is broken to a finite index subgroup, namely the subgroup of $Sl(2,\mathbb{Z})$ leaving all (h^i,g^i) invariant modulo 2. It will be quite useful to have a precise understanding of these subgroups, to which we now turn.

A.2 Congruence 2 subgroups of $Sl(2,\mathbb{Z})$

Under the modular group $Sl(2,\mathbb{Z})$, the characteristics (h,g) transform as a doublet. The subgroup of $Sl(2,\mathbb{Z})$ leaving (h,g) invariant modulo 2 is easily found to be

$$\left[{h \atop g}\right]=\left[{1 \atop 0}\right] \ : \qquad \Gamma_2^+ := \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \ , \quad \left\{ \begin{array}{c} T^2 : \tau \to \tau+2 \\ STS : \tau \to \tau/(1-\tau) \end{array} \right. \ , \tag{A.6a}$$

$$\left[{h \atop g}\right]=\left[{0 \atop 1}\right] \ : \qquad \Gamma_2^- := \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \ , \quad \left\{ \begin{array}{c} T : \tau \to \tau+1 \\ ST^2S : \tau \to \tau/(1-2\tau) \end{array} \right. \ , \tag{A.6b}$$

$$\left[{h \atop g}\right]=\left[{1 \atop 1}\right] \ : \qquad \Gamma_2^0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \left\{ \begin{array}{l} T^2 : \tau \to \tau+2 \\ \ S : \tau \to -1/\tau \end{array} \right. \ , \tag{A.6c}$$

where we represented the subgroups by the value of the allowed matrices modulo 2 (where $*$ stands for 0 or 1), and listed their generators. These three subgroups are of index 3 in $Sl(2,\mathbb{Z})$, and correspond to the invariance groups (modulo phases and weights) of $\vartheta_4, \vartheta_2, \vartheta_3$ respectively. Equivalently, they are the invariance subgroups of $Z(\tau/2), Z(2\tau), Z((\tau+1)/2)$ respectively, where Z is an $Sl(2,\mathbb{Z})$ modular form. The intersection of any two of these subgroups gives the index 6 subgroup of $Sl(2,\mathbb{Z})$

$$\Gamma_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \left\{ \begin{array}{c} T^2 : \tau \to \tau+2, \\ ST^2S : \tau \to \tau/(1-2\tau) \end{array} \right. \ . \tag{A.7}$$

Therefore, for several non-vanishing shifts (h^i, g^i), the unbroken group is either $\Gamma_2^{-,0,+}$ if all the (h^i, g^i) are the same, or Γ_2 if they are different. The lattice sums (A.3) hence either form a length-3 orbit in the first case, or a length-6 orbit in the second.

The fundamental domains $\mathcal{F}_2^{+,0,-}$ of the upper-half-plane for the groups $\Gamma_2^{-,0,+}$ are three-fold and six-fold covers respectively of the fundamental domain $\mathcal{F}$ of $Sl(2,\mathbb{Z})$. Integral over these fundamental domains can be converted into each other at the expense of introducing appropriate orbits. In particular, we have, for a Γ_2^- modular invariant function Φ,

$$\int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2} \Phi(\tau) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left[\Phi(\tau) + \Phi\left(-\frac{1}{\tau}\right) + \Phi\left(-\frac{1}{\tau+1}\right) \right] \tag{A.8}$$

and for an $Sl(2,\mathbb{Z})$ modular invariant function Z,

$$\begin{aligned} &\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left[Z(2\tau) + Z\left(\frac{\tau}{2}\right) + Z\left(\frac{\tau+1}{2}\right) \right] = \\ &\qquad = \int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2} Z(2\tau) = \int_{\mathcal{F}_2^+} \frac{d^2\tau}{\tau_2^2} Z\left(\frac{\tau}{2}\right) = \int_{\mathcal{F}_2^0} \frac{d^2\tau}{\tau_2^2} Z\left(\frac{\tau+1}{2}\right) \ . \end{aligned} \tag{A.9}$$

Moreover, by changing integration variables to $\rho = 2\tau$, this can yet be rewritten as

$$\int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2} Z(2\tau) = \int_{\mathcal{F}_2^+} \frac{d^2\rho}{\rho_2^2} Z(\rho) = 3 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(2\tau) \ . \tag{A.10}$$

A.3 Summation identities

Since the string world-sheet theory is modular invariant, the shifted sums (A.1) have to appear in modular invariant combinations. These combinations amount to projecting the original unshifted partition function $Z_{d,d}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ to even momenta or add half-integer winding sectors. As a result, they can be re-expressed as unshifted partition functions of tori with different moduli. In particular,

$$\sum_{\mathrm{d},\mathrm{d}'} Z_{d,d}\begin{bmatrix}\mathrm{d}'\\ \mathrm{d}\end{bmatrix}(g,b;\tau) = 2^d Z_{d,d}(g/4,b/4;\tau)\ . \tag{A.11}$$

In particular, for $d=1$ we have

$$\frac{1}{2}\left(Z_{1,1}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](R) + Z_{1,1}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](R) + Z_{1,1}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](R) + Z_{1,1}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](R)\right) = Z_{1,1}(R/2)\ . \tag{A.12}$$

For $d=2$, we will also need the following identities (see for instance [16]):

$$\frac{1}{2}\left(Z\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}10\\10\end{smallmatrix}\right]\right) = Z(T/2,2U)\ , \tag{A.13a}$$

$$\frac{1}{2}\left(Z\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]\right) = Z(T/2,U/2)\ , \tag{A.13b}$$

$$\frac{1}{2}\left(Z\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}00\\11\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right]\right) = Z(T/2,(U+1)/2)\ , \tag{A.13c}$$

$$\begin{aligned}\frac{1}{2}\Big(Z\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right] &+ Z\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}01\\11\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}10\\11\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}11\\01\end{smallmatrix}\right] + Z\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right]\Big) = \\ &2Z(T/4,U) - Z(T/2,2U) - Z(T/2,U/2) - Z(T/2,(U+1)/2) + Z(T,U)\ .\end{aligned} \tag{A.13d}$$

We also note the partial sums, valid for any d,

$$Z_{d,d}\left[\begin{smallmatrix}0\\ \mathrm{d}\end{smallmatrix}\right](\tau) = 2^{d/2} Z_{d,d}\left(g/2,b/2;2\tau\right)\ , \tag{A.14a}$$

$$Z_{d,d}\left[\begin{smallmatrix}\mathrm{d}\\0\end{smallmatrix}\right](\tau) = 2^{d/2} Z_{d,d}\left(g/2,b/2;\frac{\tau}{2}\right)\ , \tag{A.14b}$$

$$Z_{d,d}\left[\begin{smallmatrix}\mathrm{d}\\ \mathrm{d}\end{smallmatrix}\right](\tau) = 2^{d/2} Z_{d,d}\left(g/2,b/2;\frac{\tau+1}{2}\right)\ , \tag{A.14c}$$

where the summation over the d-digit numbers d is implicit. This shows that the three sums in (A.14) form a length-3 orbit of $Sl(2,\mathbb{Z})$.

A.4 Lattice integral on extended fundamental domain

We now would like to evaluate modular integrals of the form

$$I_{d,d}[\Phi] = \int_{\mathcal{F}_2^-} \frac{d^2\tau}{\tau_2^2} Z_{d,d} \left[{}^{0}_{\rm d} \right] (\tau, \bar\tau) \Phi(\bar\tau) \ , \tag{A.15}$$

where $\Phi(\tau)$ is an almost holomorphic form invariant under the index 2 subgroup Γ_2^- of $Sl(2,\mathbb{Z})$, a typical example being $\Phi(\tau) = (\alpha E_4 + \beta \hat{E}_2^2) \vartheta_3^8 \vartheta_4^8 / \eta^{24}$. The sum over $\mathrm{d} = 0 \dots 2^p - 1$ is implicit, and we shall focus here on $p = d$, even though many of the results can be extended to the less symmetric case $p < d$. For $\Phi = 1, d = 2$, this integral has been computed in [35] and later in [65] by a different method. For $\Phi \neq 1$ and $d = 2$, the basic observations have been made in [23], and we will streamline and greatly extend their result to all d.

In order to compute this integral, we first convert the shifted lattice sum $Z \left[{}^{0}_{\rm d} \right]$ into a standard unshifted sum using (A.14), and then change variables to $\rho = 2\tau$ as in (A.10). We obtain

$$I_{d,d}[\Phi] = 2^{d/2} \int_{\mathcal{F}_2^+} \frac{d^2\rho}{\rho_2^2} Z(g/2, b/2, \rho) \Phi\left(\frac{\bar\rho}{2}\right) \ . \tag{A.16}$$

We then unfold the integral on the extended fundamental domain $\mathcal{F}_2^+$ into an integral on the fundamental domain of $Sl(2,\mathbb{Z})$:

$$I_{d,d}[\Phi] = 2^{d/2} \int_{\mathcal{F}} \frac{d^2\rho}{\rho_2^2} Z(g/2, b/2, \rho) \left[\Phi\left(\frac{\bar\rho}{2}\right) + \Phi\left(-\frac{1}{2\bar\rho}\right) + \Phi\left(\frac{\bar\rho + 1}{2}\right) \right] \ . \tag{A.17}$$

Using the definition of the Hecke operator on a Γ_2^- modular form of weight w,

$$H_{\Gamma_2^-} \cdot \Phi(\tau) = \frac{1}{2} \left(\tau^{-w} \Phi\left(-\frac{1}{2\tau}\right) + \Phi\left(\frac{\tau}{2}\right) + \Phi\left(\frac{\tau+1}{2}\right) \right) \ , \tag{A.18}$$

we recognize in (A.17) the action of this operator on the modular form Φ:

$$I_{d,d}[\Phi] = 2^{\frac{d}{2}+1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(g/2, b/2; \tau) \ H_{\Gamma_2^-} \cdot \Phi(\bar\tau) \ . \tag{A.19}$$

This operator maps Γ_2^- modular forms into $Sl(2,\mathbb{Z})$ modular forms and preserves the weight. $H_{\Gamma_2^-} \cdot \Phi$ is therefore an almost holomorphic form of $Sl(2,\mathbb{Z})$ of zero weight, so that (A.19) is well defined. We can now use the standard techniques to express this integral as a sum over zero, degenerate and non-degenerate orbits. A great simplification comes from the fact under suitable assumptions, the image of Φ under the Hecke operator has no pole, and *has therefore to be a constant* λ [35]. The relevant constants are listed in Appendix B.3. This observation is at the heart of the

simplifications that allow the heterotic-type II duality to work. In that case, we can thus rewrite (A.17) as

$$I_{d,d}[\Phi] = 2^{\frac{d}{2}+1}\lambda \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z(g/2, b/2; \tau) \ . \tag{A.20}$$

This is now a standard integral $I_d = \int Z_{d,d}(g, b, \tau)$ over the fundamental domain of $Sl(2,\mathbb{Z})$, which can be for instance represented in terms of Eisenstein series [10]. For $d = 1, 2, 3, 4$, we recall in particular

$$I_1(R) = \frac{\pi}{3}\left(R + \frac{l_s^2}{R}\right) \ , \tag{A.21a}$$

$$I_2(T,U) = -\log \frac{8\pi e^{1-\gamma_E}}{3\sqrt{3}} T_2 U_2 |\eta(U)|^4 |\eta(T)|^4 \ , \tag{A.21b}$$

$$I_3(g,b) = \frac{1}{\pi}\ \mathcal{E}^{SO(3,3,\mathbb{Z})}_{\mathbf{4};s=1} = \frac{1}{\pi}\ \mathcal{E}^{Sl(4,\mathbb{Z})}_{\mathbf{4};s=1} \ , \tag{A.21c}$$

$$I_4(g,b) = \frac{1}{\pi}\ \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{V};s=1} = \frac{1}{\pi}\ \mathcal{E}^{SO(4,4,\mathbb{Z})}_{\boldsymbol{C};s=1} \ , \tag{A.21d}$$

where the normalization here differs from that of [10]. For a given discrete duality symmetry group $G(\mathbb{Z})$, the order s Eisenstein series of representation $\mathcal{R}$ is defined by,

$$\mathcal{E}^{G(\mathbb{Z})}_{\boldsymbol{\mathcal{R}};s} = \sum_{m\in\Lambda_{\mathcal{R}}\backslash\{0\}} \delta(m\wedge m)\left[\mathcal{M}^2(\mathcal{R})\right]^{-s} \tag{A.22}$$

where we refer to [10] for explicit expressions of the $G(\mathbb{Z})$-invariant BPS masses $\mathcal{M}^2(\mathcal{R})$ and the half-BPS condition $m \wedge m = 0$, that are relevant for the cases in (A.21c), (A.21d).

The simplification that occurred in the computation of (A.15) is actually of much more general validity, and would hold provided the shifted lattice sum can be rewritten as $Z(2\tau)$ for some modular invariant function τ. The insertion Φ can then be replaced by its value, *when constant*, under the Hecke operator $H_{\Gamma_2^-}$:

$$I_{d,d}[\Phi] = \frac{2\lambda}{3} I_{d,d}[1] \quad \text{if} \quad H_{\Gamma_2^-}(\Phi) = \lambda \ . \tag{A.23}$$

The same also holds for fractional shifts $1/n$, $n > 2$ that occur in $\mathbb{Z}_n$ orbifolds, although we will not explore this topic. We also mention that the rule (A.23) holds as well in the presence of insertions of momenta p_R^i, p_L^i.

B. Useful modular identities

We refer to Appendix F of [66] for generalities and useful identities on modular forms. Here we list the modular identities that are useful for the present work.

B.1 Theta functions and their derivatives

Our conventions for the Jacobi Theta functions are

$$\vartheta\left[{a \atop b}\right](v,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n-\frac{a}{2}\right)^2} e^{(v-i\pi b)\left(n-\frac{a}{2}\right)} \ , \quad q = e^{2\pi i\tau} \ , \tag{B.1}$$

where the normalization of v is non-standard. We also use the Erderlyi notation

$$\vartheta_1 = \vartheta\left[{1 \atop 1}\right] \ , \ \vartheta_2 = \vartheta\left[{1 \atop 0}\right] \ , \ \vartheta_3 = \vartheta\left[{0 \atop 0}\right] \ , \ \vartheta_4 = \vartheta\left[{0 \atop 1}\right] \ . \tag{B.2}$$

Note that $\vartheta_{2,3,4}$ are even functions of their argument v while ϑ_1 is odd, and $\vartheta_1' = -i\eta^3$ where we denote by a prime the differentiation with respect to v. For more than one derivation $\partial/\partial v$, the result is not modular covariant anymore, and has to be corrected by non-holomorphic contributions, analogous to the replacement $E_2 \to \hat{E}_2 = E_2 - 3/(\pi\tau_2)$. We use the multiprime symbols for the result of this covariantization. We hence have

$$\frac{\vartheta_2''}{\vartheta_2} = \frac{1}{12}\left(\hat{E}_2 + \vartheta_3^4 + \vartheta_4^4\right) \tag{B.3a}$$

$$\frac{\vartheta_3''}{\vartheta_3} = \frac{1}{12}\left(\hat{E}_2 + \vartheta_2^4 - \vartheta_4^4\right) \tag{B.3b}$$

$$\frac{\vartheta_4''}{\vartheta_4} = \frac{1}{12}\left(\hat{E}_2 - \vartheta_2^4 - \vartheta_3^4\right) \tag{B.3c}$$

$$\frac{\vartheta_2''''}{\vartheta_2} = \frac{1}{48}\left(-2E_4 + \hat{E}_2^2 + 2\hat{E}_2(\vartheta_3^4 + \vartheta_4^4) + 3\vartheta_2^8\right) \tag{B.3d}$$

$$\frac{\vartheta_3''''}{\vartheta_3} = \frac{1}{48}\left(-2E_4 + \hat{E}_2^2 + 2\hat{E}_2(\vartheta_2^4 - \vartheta_4^4) + 3\vartheta_3^8\right) \tag{B.3e}$$

$$\frac{\vartheta_4''''}{\vartheta_4} = \frac{1}{48}\left(-2E_4 + \hat{E}_2^2 - 2\hat{E}_2(\vartheta_2^4 + \vartheta_3^4) + 3\vartheta_4^8\right) \ . \tag{B.3f}$$

The following combinations will be particularly relevant

$$\frac{\vartheta_3''''}{\vartheta_3} + \frac{\vartheta_4''''}{\vartheta_4} - 3\left(\frac{\vartheta_3''}{\vartheta_3}\right)^2 - 3\left(\frac{\vartheta_4''}{\vartheta_4}\right)^2 = -\frac{1}{8}\vartheta_2^8 \tag{B.4a}$$

$$\left(\frac{\vartheta_3''}{\vartheta_3}\right)^2 + \left(\frac{\vartheta_4''}{\vartheta_4}\right)^2 - 2\frac{\vartheta_3''}{\vartheta_3}\frac{\vartheta_4''}{\vartheta_4} = \frac{1}{16}\vartheta_2^8 \tag{B.4b}$$

and the following identities are useful to make contact with [23]:

$$\left(\frac{\vartheta_3''}{\vartheta_3}\right)^2 + \left(\frac{\vartheta_4''}{\vartheta_4}\right)^2 = \frac{1}{72}\left(\hat{E}_2 - \frac{\vartheta_3^4 + \vartheta_4^4}{2}\right)^2 + \frac{1}{32}\vartheta_2^8 \tag{B.5a}$$

$$2\frac{\vartheta_3''}{\vartheta_3}\frac{\vartheta_4''}{\vartheta_4} = \frac{1}{72}\left(\hat{E}_2 - \frac{\vartheta_3^4 + \vartheta_4^4}{2}\right)^2 - \frac{1}{32}\vartheta_2^8 \ . \tag{B.5b}$$

B.2 Summation identities

In our “modular Einstein convention”, $\alpha = 2,3,4$ is summed over all even spin structures. The following equations are useful to convert the contribution of the unshifted orbit into a sum of shifted orbits:

$$\vartheta_\alpha^{16} - \left[2\vartheta_3^8\vartheta_4^8 + \text{ orb.}\right] = 0 \tag{B.6a}$$

$$\vartheta''_\alpha\ \vartheta_\alpha^{15} - \left[\vartheta_3^8\vartheta_4^8\left(\frac{\vartheta''_3}{\vartheta_3} + \frac{\vartheta''_4}{\vartheta_4}\right) + \text{ orb.}\right] = 0 \tag{B.6b}$$

$$\vartheta''''_\alpha\ \vartheta_\alpha^{15} - \left[\vartheta_3^8\vartheta_4^8\left(\frac{\vartheta''''_3}{\vartheta_3} + \frac{\vartheta''''_4}{\vartheta_4}\right) + \text{ orb.}\right] = 96\eta^{24} \tag{B.6c}$$

$$(\vartheta''_\alpha)^2\ \vartheta_\alpha^{14} - \left[2\vartheta_3^8\vartheta_4^8\frac{\vartheta''_3}{\vartheta_3}\frac{\vartheta''_4}{\vartheta_4} + \text{ orb.}\right] = 16\eta^{24} \tag{B.6d}$$

$$(\vartheta''_\alpha)^2\ \vartheta_\alpha^{14} - \left[\vartheta_3^8\vartheta_4^8\left(\left(\frac{\vartheta''_3}{\vartheta_3}\right)^2 + \left(\frac{\vartheta''_4}{\vartheta_4}\right)^2\right) + \text{ orb.}\right] = -32\eta^{24}\ . \tag{B.6e}$$

where + orb. denotes the two extra terms obtained from the first by applying S and ST modular transformations.

B.3 Hecke identities

As proven in Appendix A.4, insertions of almost holomorphic modular forms into integrals of projected lattice sums can be replaced by two-thirds their value λ under the Hecke operator (A.18). Here we list the corresponding value for the modular forms of interest

$$H_{\Gamma_2^-}\left[\vartheta_3^8\vartheta_4^8/\eta^{24}\right] = 0\ , \quad H_{\Gamma_2^-}\left[\vartheta_3^8\vartheta_4^8\hat{E}_2/\eta^{24}\right] = 0 \tag{B.7a}$$

$$H_{\Gamma_2^-}\left[\vartheta_3^8\vartheta_4^8E_4/\eta^{24}\right] = 360\ , \quad H_{\Gamma_2^-}\left[\vartheta_3^8\vartheta_4^8\hat{E}_2^2/\eta^{24}\right] = 72 \tag{B.7b}$$

$$H_{\Gamma_2^-}\left[\frac{\vartheta_3^8\vartheta_4^8}{\eta^{24}}\left(\frac{\vartheta''_3}{\vartheta_3} + \frac{\vartheta''_4}{\vartheta_4}\right)\right] = 0 \tag{B.7c}$$

$$H_{\Gamma_2^-}\left[\frac{\vartheta_3^8\vartheta_4^8}{\eta^{24}}\left(\frac{\vartheta''_3}{\vartheta_3} + \frac{\vartheta''_4}{\vartheta_4}\right)\hat{E}_2\right] = 24 \tag{B.7d}$$

$$H_{\Gamma_2^-}\left[\frac{\vartheta_3^8\vartheta_4^8}{\eta^{24}}\left(\frac{\vartheta_3''''}{\vartheta_3}+\frac{\vartheta_4''''}{\vartheta_4}\right)\right]=0 \tag{B.7e}$$

$$H_{\Gamma_2^-}\left[\frac{\vartheta_3^8\vartheta_4^8}{\eta^{24}}\left(\left(\frac{\vartheta_3''}{\vartheta_3}\right)^2+\left(\frac{\vartheta_4''}{\vartheta_4}\right)^2\right)\right]=16 \tag{B.7f}$$

$$H_{\Gamma_2^-}\left[\frac{\vartheta_3^8\vartheta_4^8}{\eta^{24}}\left(\frac{\vartheta_3''}{\vartheta_3}\right)\left(\frac{\vartheta_4''}{\vartheta_4}\right)\right]=-4\ . \tag{B.7g}$$

These formulae can be obtained by looking at the leading q expansions, or by using the following duplication identities:

$$\vartheta_2(2\tau)=\frac{1}{\sqrt{2}}\sqrt{\vartheta_3^2(\tau)-\vartheta_4^2(\tau)}\ ,\quad \vartheta_3(2\tau)=\frac{1}{\sqrt{2}}\sqrt{\vartheta_3^2(\tau)+\vartheta_4^2(\tau)} \tag{B.8a}$$

$$\vartheta_4(2\tau)=\sqrt{\vartheta_3(\tau)\vartheta_4(\tau)}\ ,\quad \eta(2\tau)=2^{-2/3}\vartheta_2^{2/3}(\tau)(\vartheta_3(\tau)\vartheta_4(\tau))^{1/6} \tag{B.8b}$$

$$\vartheta_2(\tau/2)=\sqrt{2\vartheta_2(\tau)\vartheta_3(\tau)}\ ,\quad \vartheta_3(\tau/2)=\sqrt{\vartheta_3^2(\tau)+\vartheta_2^2(\tau)} \tag{B.8c}$$

$$\vartheta_4(\tau/2)=\sqrt{\vartheta_3^2(\tau)-\vartheta_2^2(\tau)}\ ,\quad \eta(\tau/2)=2^{-1/6}\vartheta_4^{2/3}(\tau)(\vartheta_2(\tau)\vartheta_3(\tau))^{1/6} \tag{B.8d}$$

$$\vartheta_2\left(\frac{\tau+1}{2}\right)=e^{\frac{i\pi}{8}}\sqrt{2\vartheta_2(\tau)\vartheta_4(\tau)}\ ,\quad \vartheta_3\left(\frac{\tau+1}{2}\right)=\sqrt{\vartheta_4^2(\tau)+i\vartheta_2^2(\tau)} \tag{B.8e}$$

$$\vartheta_4\left(\frac{\tau+1}{2}\right)=\sqrt{\vartheta_4^2(\tau)-i\vartheta_2^2(\tau)}\ ,\quad \eta\left(\frac{\tau+1}{2}\right)=2^{-1/6}\ e^{\frac{i\pi}{24}}\ \vartheta_3^{2/3}(\tau)(\vartheta_2(\tau)\vartheta_4(\tau))^{1/6} \tag{B.8f}$$

$$\vartheta_2(\tau)=2\frac{\eta^2(2\tau)}{\eta(\tau)}\ ,\quad \vartheta_4(\tau)=\frac{\eta^2(\tau/2)}{\eta(\tau)}\ ,\quad \vartheta_3(\tau)=e^{i\pi/12}\frac{\eta^2((\tau+1)/2)}{\eta(\tau)} \tag{B.8g}$$

$$\eta(2\tau)\ \eta(\tau/2)\ \eta\left((\tau+1)/2\right)=e^{-i\pi/24}\eta^3(\tau)\ . \tag{B.8h}$$

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